Persistence of randomness in the macroscopic limit.
I Spinodal decomposition.

Errico Presutti

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Small fluctuations are produced which may become macroscopic *after long times due to instabilities of the macroscopic equations.*

When a system is removed far from equilibrium, it will often undergo a transition from a spatially uniform state to a state with spatial variations, referred to as patterns.

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Random spatial patterns appear in the course of the

Spinodal decomposition.

Phase diagram of a ferromagnet

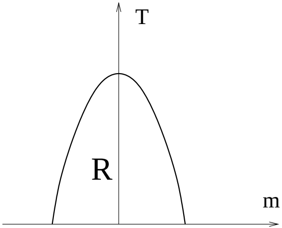

Figure: Temperature versus magnetization, magnetizations in the region R are "forbidden"

Any magnetization outside R can be realized in a thermodynamic equilibrium state by applying suitable external magnetic field. There is no pure phase with magnetization in R

Time 0^-: $T > T_c$ and no external magnetic field, $m = 0$.
Time 0^+: Fast cooling drives system to $T < T_c$ with still $m = 0$.

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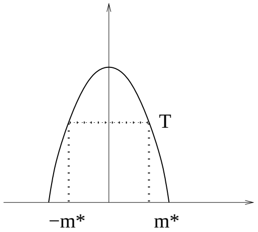
m^* and $-m^*$ equilibrium magnetizations at $T < T_c$ and $h = 0$.



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$t = \infty$: Formation of a single phase (non conservative evolutions), or Wulff shape (conserved evolution).
Wulff shape is the surface with minimal surface tension dividing regions of equal volume.

(some) References.

Glauber (non conservative) dynamics in Ising with Kac potentials, first stage of decomposition:

A. De Masi, E.Orlandi, E.Presutti, L.Triolo: Glauber evolution with Kac potentials II. Fluctuations. *Nonlinearity*. **9**, 27–51 (1996).

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Cahn-Hilliard (conservative) dynamics with random initial data, early stage:
S. Maier-Paape, T. Wanner: Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions. Nonlinear dynamics. *Arch. Ration. Mech. Anal.* **151**, 187–219 (2000).

Coarsening (in conservative dynamics):
N.D. Alikakos, G. Fusco, G. Karali: Ostwald ripening in two dimensions, the rigorous derivation of the equations from the Mullins-Sekerka dynamics. *J. Differential Equations* **205**, 1–49 (2004).
B. Niethammer, J.J.L. Velzquez: On the convergence to the smooth self-similar solution in the LSW model. *Indiana Univ. Math. J.* **55**, 761–794 (2006).

Spinodal decomposition in a mean field model.

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Mean field Hamiltonian: h external magnetic field,

$$H_{h,\Lambda}(\sigma_\Lambda) = (-\frac{1}{2} m^2 - hm)|\Lambda|$$

Recalls from thermodynamics

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- For each β, the set of values of the magnetization (in a pure phase) is $\{D_h^{\pm} P(\beta, h), h \in \mathbb{R}\}$.

- When $D_h^{-} P(\beta, h) = D_h^{+} P(\beta, h) =: D_h P(\beta, h)$ there is a unique phase with magnetization $D_h P(\beta, h)$.
When $D_h^{-} P(\beta, h) < D_h^{+} P(\beta, h)$ there are two phases with magnetization $m_\beta^{\pm}(h) = D_h^{\pm} P(\beta, h)$.

Gibbs hypothesis.

The Gibbs thermodynamic pressure $P(\beta, h)$ *is:*

$$P(\beta, h) = \lim_{|\Lambda| \to \infty} \frac{1}{\beta |\Lambda|} \log Z(\beta, h, \Lambda)$$

where the "partition function" $Z(\beta, h, \Lambda)$ is

$$Z(\beta, h, \Lambda) = \sum_{\sigma_\Lambda \in \{-1,1\}^\Lambda} e^{-\beta H_{h,\Lambda}(\sigma_\Lambda)}$$

Theorem. $P(\beta, h)$ is well defined and

$$P(\beta, h) = \max_{m \in [-1,1]} \{hm - \big(-\frac{m^2}{2} - \frac{S(m)}{\beta} \big)\}$$

$$S(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}$$

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Thus $P(\beta, h)$ is the Legendre transform of

$$F(\beta, m) = -\frac{m^2}{2} - \frac{S(m)}{\beta}$$

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• $P(\beta, h)$ is a convex function of h (being the Legendre transform of $F(\beta, m)$).

• For each h there exists a highest line of slope h below the graph of $F(\beta, m)$.

• Its intersection with the graph has a minimal and maximal abscissa, $m_{\pm}(\beta, h)$, and

$$D_h^{\pm} P(\beta, h) = m_{\pm}(\beta, h)$$

Thus the value m is allowed if the tangent at m of $F(\beta, m)$ is below the graph of $F(\beta, m)$ and:

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the others are all either to the left or to the right of m.

When $\beta < 1$, $F(\beta, m)$ is strictly convex, $F''(\beta, m) > 0$,

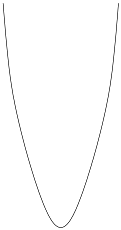

Figure: The graph of $F(\beta, m)$ for $\beta \leq 1$.

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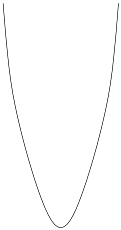

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all tangents to $F(\beta, m)$ are below the graph with single intersection, all values of $m \in [-1, 1]$ are allowed, $P(\beta, h)$ is differentiable at all h.

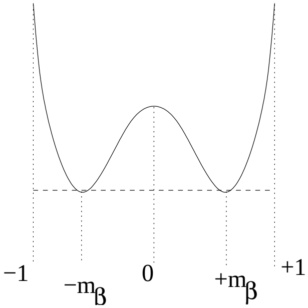

Figure: The graph of $F(\beta, m)$ for $\beta > 1$; the value at $m = \pm 1$ is $-1/2$, at $m = 0$ is $-\log 2/\beta$. The dashed line is the highest line with slope 0 below the graph, the abscissa of the intersections are $\pm m_\beta$.

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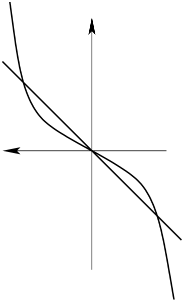

Figure: Graph of $\tanh \beta m$, $\beta > 1$: the intersections with the diagonal are at $\pm m_\beta$.

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$P(\beta, h)$ is differentiable at all $h \neq 0$, at $h = 0$

$$D_h^{\pm} P(\beta, h)\Big|_{h=0} = \pm m_\beta$$

Glauber dynamics.

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$$Lf(\sigma_\Lambda) = \sum_{x \in \Lambda} c(x, \sigma_\Lambda)\Big(f(\sigma_\Lambda^x) - f(\sigma_\Lambda)\Big), \quad c(x, \sigma_\Lambda) > 0$$

$$\sigma_\Lambda^x(y) = \begin{cases} \sigma_\Lambda(y) & \text{if} y \neq x \\ -\sigma_\Lambda(x) & \text{if} y = x \end{cases}$$

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e^{Lt} is defined by a power series expansion which converges because L is bounded.

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$c(x, \sigma_\Lambda)$ is the intensity of flipping the spin at x: $\sigma_\Lambda \to \sigma_\Lambda^x$.

Theorem. For any σ_Λ and $t \geq 0$ there is a probability measure μ on $\{-1, 1\}^\Lambda$ such that

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$$\mu(\sigma'_\Lambda) = [e^{Lt}\mathbf{1}_{\sigma'_\Lambda}](\sigma_\Lambda)$$

Glauber dynamics. When $c(x, \sigma_\Lambda)$ has the form

$$c(x, \sigma_\Lambda) = c_0(x, \sigma_{\Lambda \setminus x})\, e^{-\frac{\beta}{2}[H_\Lambda(\sigma_\Lambda^x) - H_\Lambda(\sigma_\Lambda)]}$$

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$$c_0(m^x) = \frac{1}{e^{-\beta(h+m^x)} + e^{\beta(h+m^x)}}$$

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To have simpler notation we will replace $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L} g(m) &= |\Lambda| \frac{1+m}{2} \frac{e^{\beta(h+m)}}{e^{\beta(h+m)} + e^{-\beta(h+m)}} \Big(g(m - \frac{2}{|\Lambda|}) - g(m) \Big) \\ &+ |\Lambda| \frac{1-m}{2} \frac{e^{-\beta(h+m)}}{e^{\beta(h+m)} + e^{-\beta(h+m)}} \Big(g(m + \frac{2}{|\Lambda|}) - g(m) \Big) \end{aligned}$$

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correct value has m^x instead of m in the Gibbs factor, the difference is bounded by $c/|\Lambda|$.

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$$\|e^{\mathcal{L}t} g(m) - g(u(t; m))\|_\infty \leq \|g'\|_\infty \Big(e^{ct} \frac{c'}{c|\Lambda|} \Big)^{1/2}$$

$$\frac{du(t; m)}{dt} = -u(t; m) + \tanh\{\beta(u(t; m) + h)\}, \quad u(0; m) = m$$

Proof. Sketch. Define $e^{\mathcal{L}t}(m, m') := [e^{\mathcal{L}t}\mathbf{1}_{m'}](m)$.

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$$\langle \big(m(t) - u(t)\big)^2 \rangle := \sum_{m' \in M_\Lambda} e^{\mathcal{L}t}(m, m')\big(m' - u(t)\big)^2$$

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$$\frac{d}{dt}\langle \big(m(t)-u(t)\big)^2 \rangle \;\; = \;\; \sum_{m' \in M_\Lambda} e^{\mathcal{L}t}(m,m')\{\mathcal{L}+\frac{\partial}{\partial t}\}\big(m'-u(t)\big)^2$$

Computations.

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$$
\begin{aligned}
\mathcal{L}m \quad = \quad & |\Lambda| \frac{1+m}{2} \; \frac{e^{\beta(h+m)}}{e^{\beta(h+m)} + e^{-\beta(h+m)}} \left(-\frac{2}{|\Lambda|}\right) \\
& + |\Lambda| \frac{1-m}{2} \; \frac{e^{-\beta(h+m)}}{e^{\beta(h+m)} + e^{-\beta(h+m)}} \left(\frac{2}{|\Lambda|}\right)
\end{aligned}
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&= -m+\tanh\{\beta(m+h)\} \\
&=: V(m)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}m^2 &= |\Lambda|\frac{1+m}{2}\ \frac{e^{\beta(h+m)}}{e^{\beta(h+m)}+e^{-\beta(h+m)}}\Big((m-\frac{2}{|\Lambda|})^2-m^2\Big) \\
&\quad +|\Lambda|\frac{1-m}{2}\ \frac{e^{-\beta(h+m)}}{e^{\beta(h+m)}+e^{-\beta(h+m)}}\Big((m+\frac{2}{|\Lambda|})^2-m^2\Big)
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&\quad +|\Lambda|\frac{1-m}{2}\ \frac{e^{-\beta(h+m)}}{e^{\beta(h+m)}+e^{-\beta(h+m)}}\Big((m+\frac{2}{|\Lambda|})^2-m^2\Big) \\
&= 2mV(m)+\frac{2}{|\Lambda|}V(m)
\end{aligned}
$$

$$\{\mathcal{L} + \frac{\partial}{\partial t}\}\big(m - u(t)\big)^2 \quad = \quad \mathcal{L}m^2 - 2u(t)\mathcal{L}m$$

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Conclusion of proof.

$$\frac{d}{dt}\langle\big(m(t)-u(t)\big)^2\rangle \quad \leq \quad \sum_{m'\in M_\Lambda} e^{\mathcal{L}t}(m,m')\{c|m'-u(t)|^2+\frac{c'}{|\Lambda|}\}$$

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$$|e^{\mathcal{L}t}g(m)-g(u(t;m))|\leq\|g'\|_\infty\Big(e^{ct}\frac{c'}{c|\Lambda|}\Big)^{1/2}$$

Fix $h = 0$, $\beta > 1$ and initial magnetization $m = 0$:
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After an initial time layer (infinitesimal in $\log |\Lambda|$-time units) the "normalized magnetization" is self similar at all $\tau < \tau_c$.

Heuristic proof.

- The linearized equation around $m = 0$ is

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• Suppose initially $m = |\Lambda|^{-1/2}$. Its (deterministic) linearized evolution is:

$$\psi(t) = e^{\alpha t}|\Lambda|^{-1/2}$$

As $\Lambda| \to \infty$, $\psi(t)$ is infinitesimal for $t < \tau_c \log|\Lambda|$ and explodes for $t > \tau_c \log|\Lambda|$.

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The Markov property.
Let $s \geq 0$ and $P_m(\cdot | m(s'), 0 \leq s' \leq s)$ the conditional probability on Ω given the trajectory till time s. Then for any $t > s$ and any $m^* \in M_\Lambda$:

$$P_m\big(m(t) = m^* | m(s'), 0 \leq s' \leq s\big) = e^{\mathcal{L}(t-s)} \mathbf{1}_{m^*}(m(s))$$

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$b(t)$ is a random variable on Ω: given $m(\cdot) \in \Omega$ we get $b(\cdot)$.

Suppose $b(\cdot)$ known and $m(0) = 0$, then we can compute $m(\cdot)$ as solution of

$$m(t) = \int_0^t V(m(s))\,ds + b(t)$$

A priori information on $b(t)$.

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which follows because $V(m) = \mathcal{L}m$ and therefore

$$b(t) = m(t) - \int_0^t \mathcal{L}m(s)\, ds$$

Doob's Theorem.

$$E(\max_{s \le t} \big(b^2(s)\big) \le 4E\big(b^2(t)\big)$$

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$|b(t)| \leq |\Lambda|^{-\theta}$ with probability $\to 1$ as $|\Lambda| \to \infty$.

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By a perturbative argument also $m(t)$ solution of the full equation has the same property.

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Analysis is based on martingale convergence theorems.