Ordering and criticality in one dimensional driven systems

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Vienna, 02-06 June, 2008

Steady states of driven systems

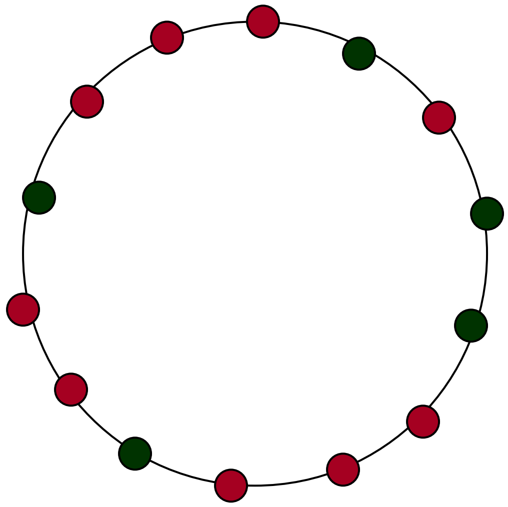
Ordering and phase separation in 1d driven systems (?)

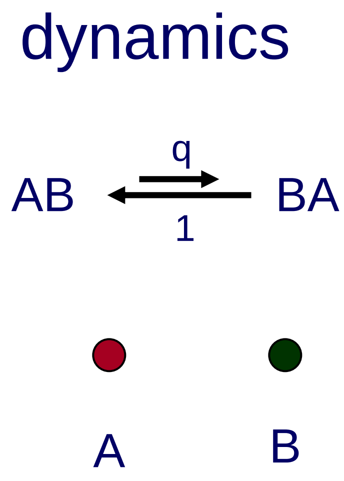
local, noisy dynamics
no detailed balance

A criterion for phase separation in such systems (?)
Types or ordering (?)

This problem is of interest e.g. in studying traffic jam models.

Minimal model: Asymmetric Simple Exclusion Process (ASEP)





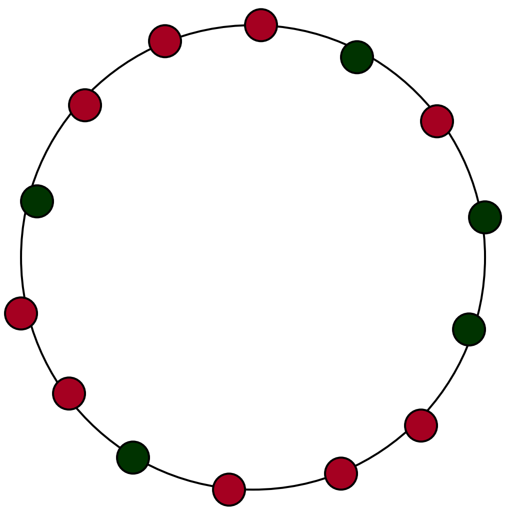
Steady State:

- q=1 corresponds to an Ising model at $T=\infty$
- All microscopic states are equally probable.
- Density is macroscopically homogeneous.

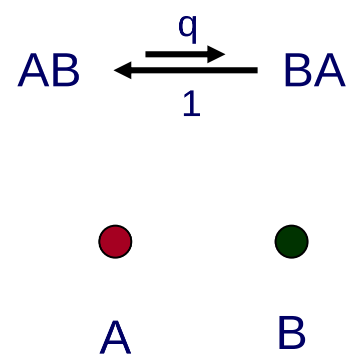
No liquid-gas transition (for any density and q).

Extensions of the model:

- more species (e.g. ABC model)
- transition rate (q) depends on local configuration (e.g. KLS model Katz, Lebowitz, Spohn)



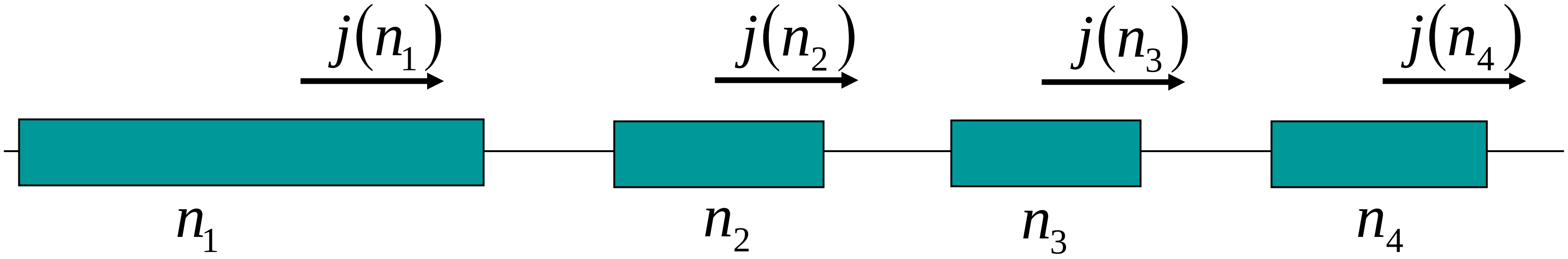
dynamics



Ordering and phase separation in such models (?)

Given a 1d driven process, does it exhibit phase separation?

Criterion for phase separation



domains exchange particles via their current $j(n)$

if $j(n)$ decreases with n: coarsening may be expected to take place.

quantitative criterion?

Phase separation takes place in one of two cases:

★ **Case A:** $j(n) \to 0$ as $n \to \infty$

e.g. $\exp(-n)$

★ **Case B:** $j_\infty \neq 0$ and

$j(n)/j_\infty - 1 \to 0$ slower than 2/n

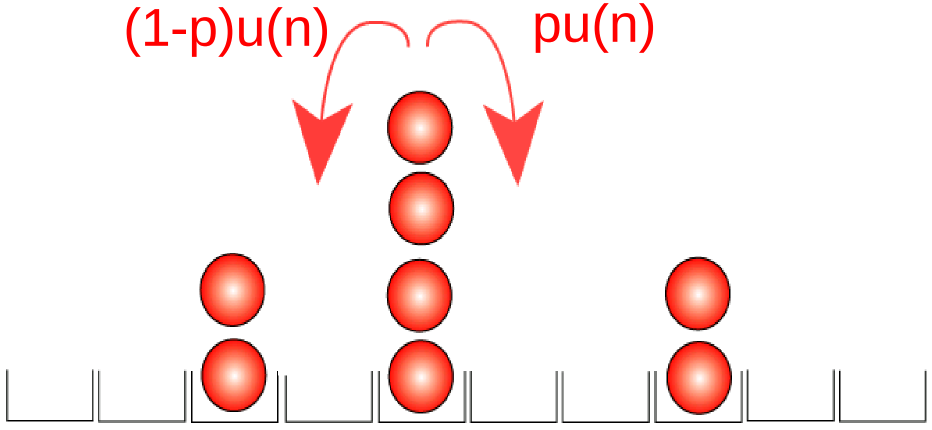
$$(\; b/n \quad b>2 \quad \text{or} \quad 1/n^s \quad s<1 \;)$$

Zero Range Process

- prototype of non-equilibrium models simple model, no detailed balance it may be used to probe non-equilibrium phenomena.

Zero Range Processes

Particles in boxes with the following dynamics:



Steady state distribution of ZRP processes:

- product measure
- $p(n) \propto \prod_{k=1}^{n} \frac{z}{u(k)}$ where z is a fugacity

For example if u(n)=u (independent of n)

$$p(n) \propto \left(\frac{z}{u}\right)^{n} \propto e^{-n/\xi}$$

with $< n >= \sum_{n} np(n)$ (determines ξ)

An interesting choice of u(n) provides a condensation transition at high densities.

$$u(n) = (1 + b/n)$$

$$p(n) \propto \prod_{k=1}^{n} \frac{z}{u(k)}$$

$$-\sum \ln u(k) = -\sum_{k=1}^{n} \ln(1 + \frac{b}{k}) \approx -b \ln n$$

$$p(n) = \frac{z^n}{n^b} = \frac{e^{-n/\xi}}{n^b}$$

The correlation length is determined by

$$\int np(n)\,dn = \langle n \rangle \qquad\qquad p(n) = \frac{e^{-n/\xi}}{n^b}$$

$\langle n \rangle$ -average number of particles in a box

- For b<2 there is always a solution with a finite ξ for any density. Thus no condensation.

$$\xi \to \infty: \qquad \int \frac{1}{n^{b-1}} dn \to \infty \qquad \text{for } b \le 2$$

$$\int np(n)\,dn = \langle n \rangle \qquad\qquad p(n) = \frac{e^{-n/\xi}}{n^b}$$

$$\xi \to \infty: \qquad \int \frac{1}{n^{b-1}}\,dn = c \qquad \text{for } b \geq 2$$

- For b>2 there is a maximal possible density, and hence a transition of the Bose Einstein Condensation type.

For densities larger than c a condensate is formed which contains a macroscopically large number of particles.

Phase diagram for b>2

$$\frac{e^{-n/\xi}}{n^b}$$

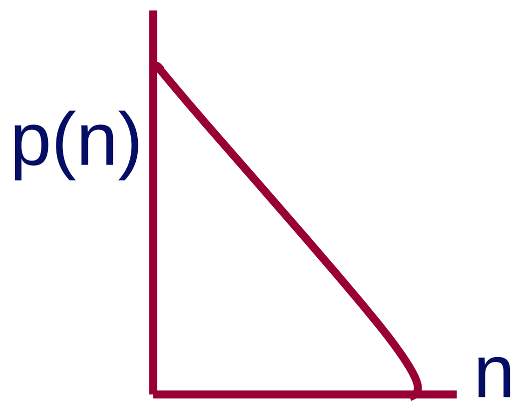
$$\frac{1}{n^b}$$

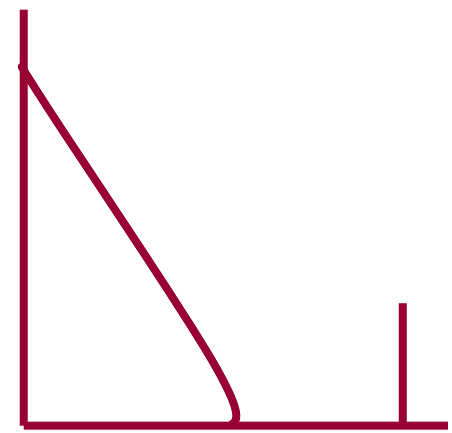
$$\frac{1}{n^b} + a\delta(n-(\rho-\rho_c)L)$$

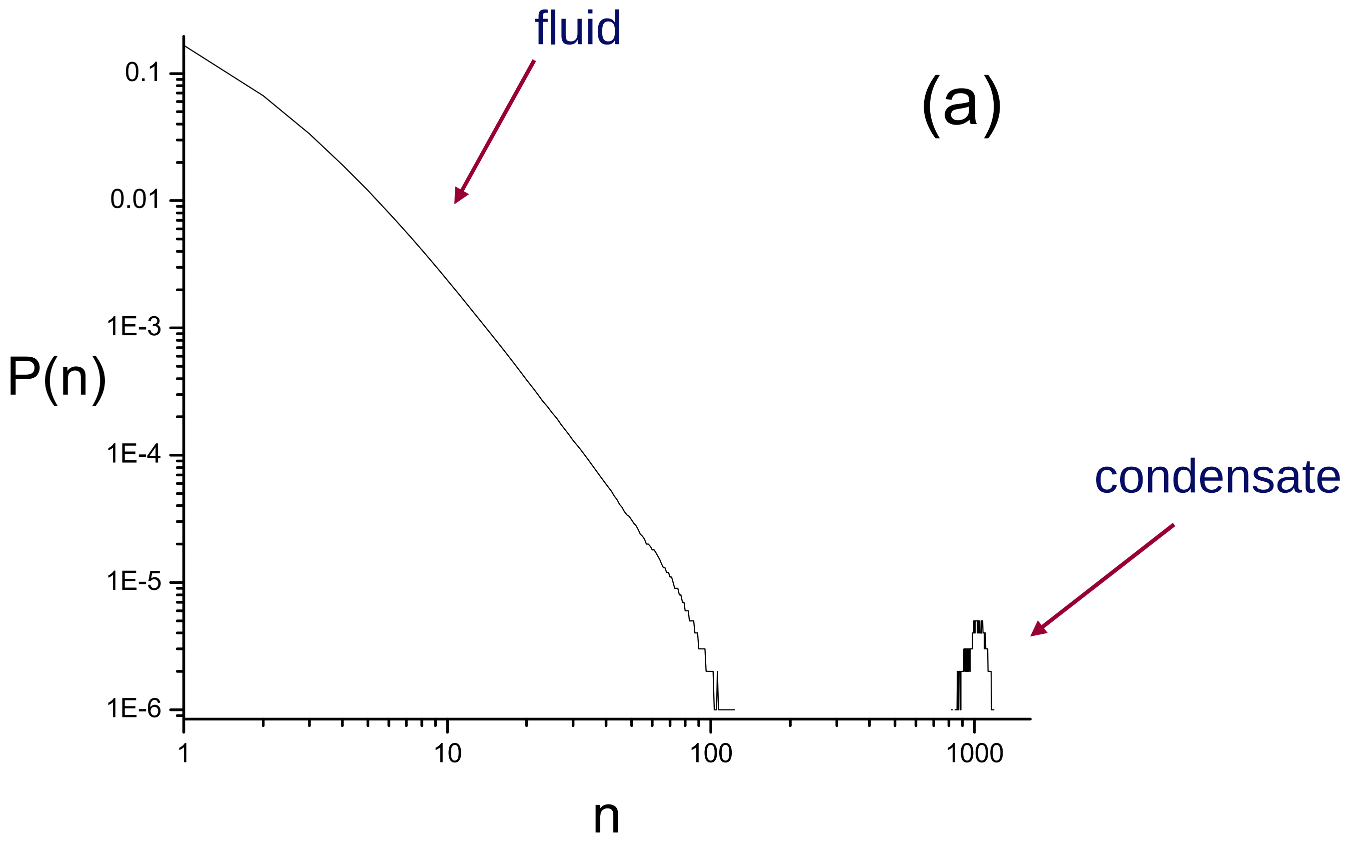
0

$$\frac{1}{b-2}$$

ρ







L-1000, N=3000, b=3

Use ZRP to probe possible types of ordering.

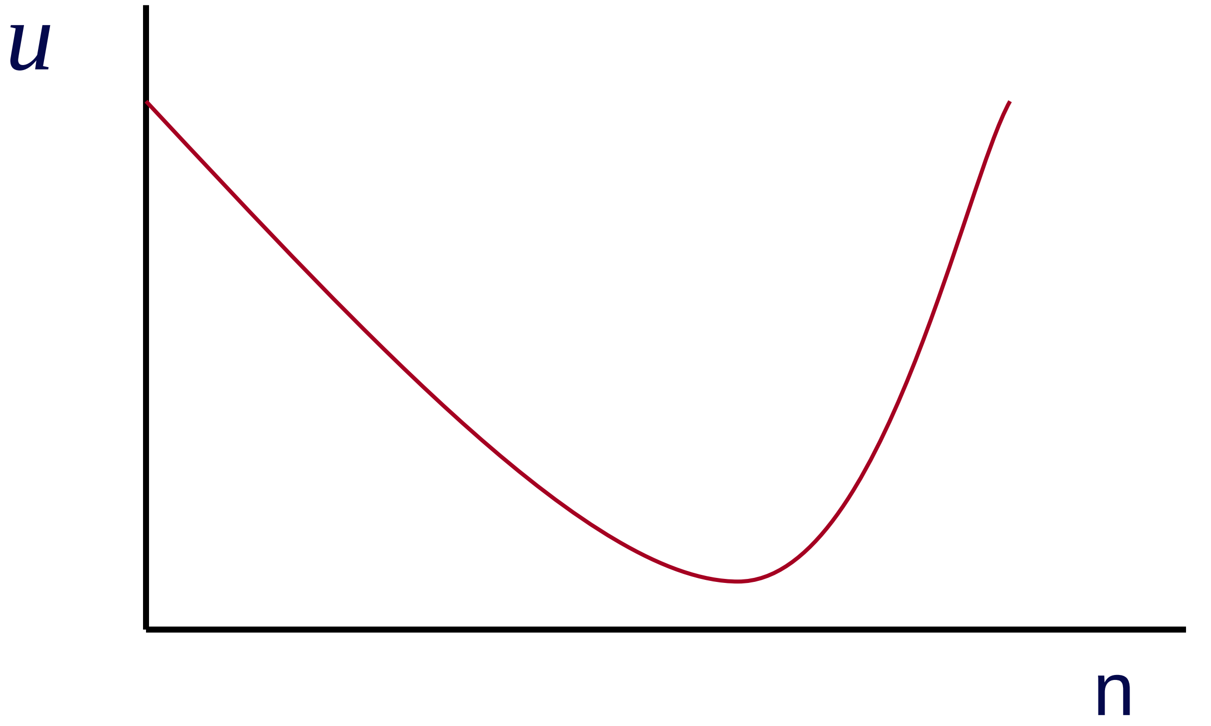
- Multiple condensates?
- Can the critical phase exist over a whole region rather than at a point in parameter space?

Non-monotonic hopping rate – multiple condensates

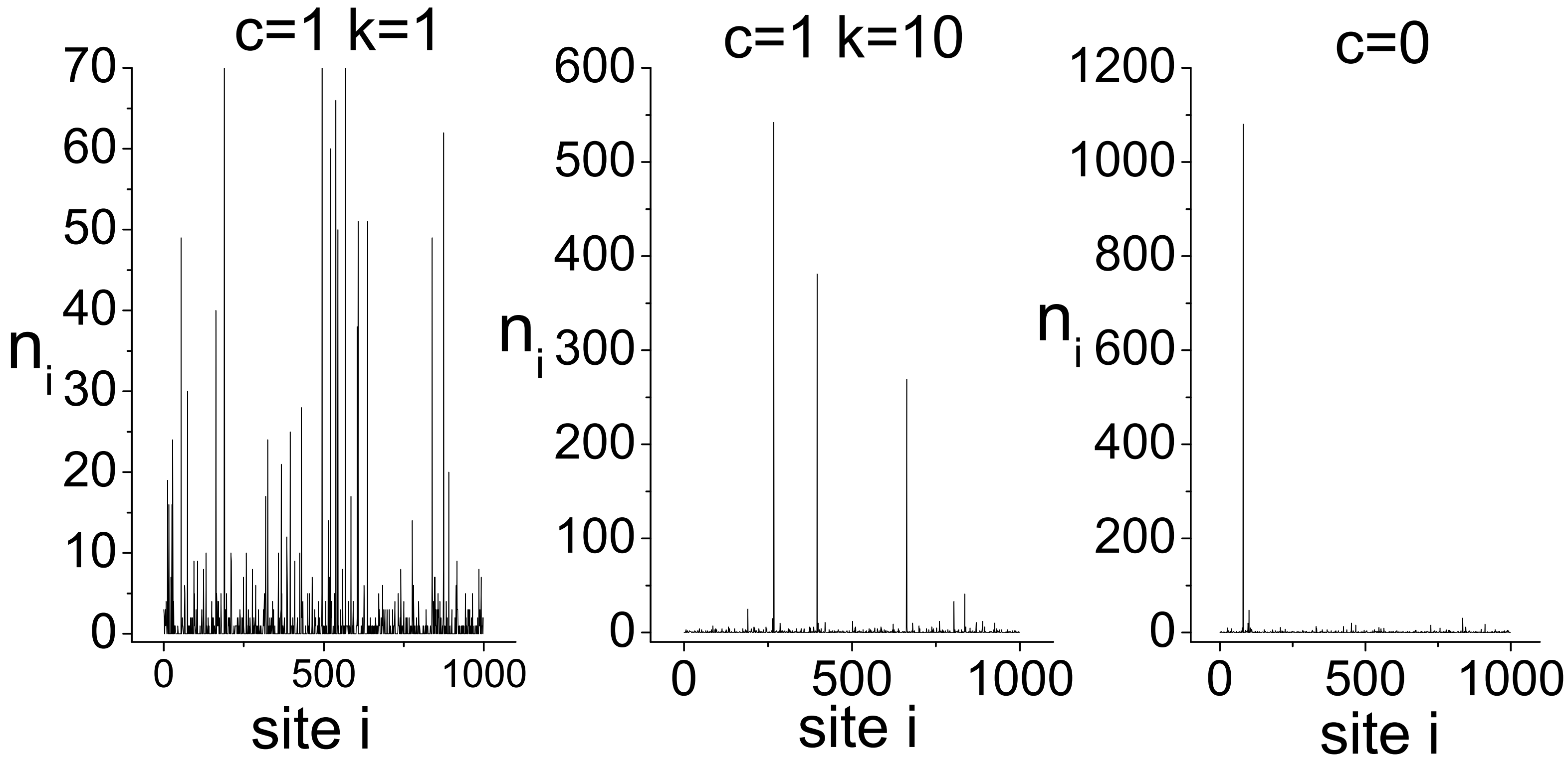
Y. Schwatrzkopf, M. R. Evans, D. Mukamel. J. Phys. A, 41, 205001 (2008)

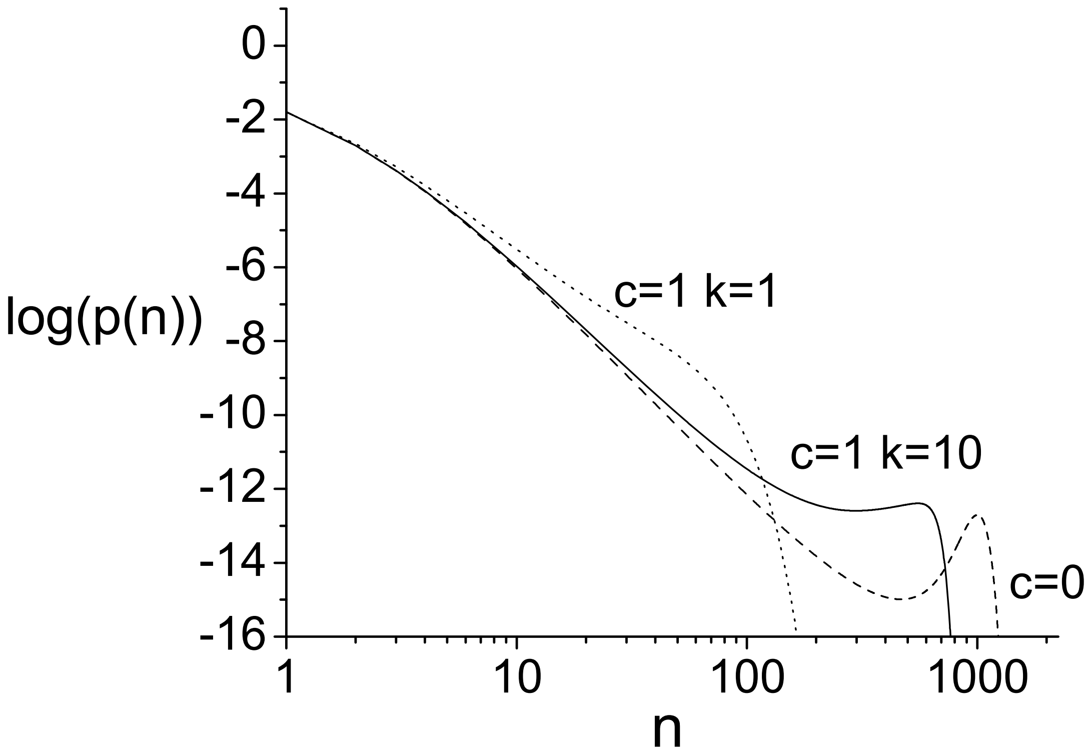
$$u(n) = 1 + \frac{b}{n} + c\left(\frac{n}{L}\right)^k$$

L sites

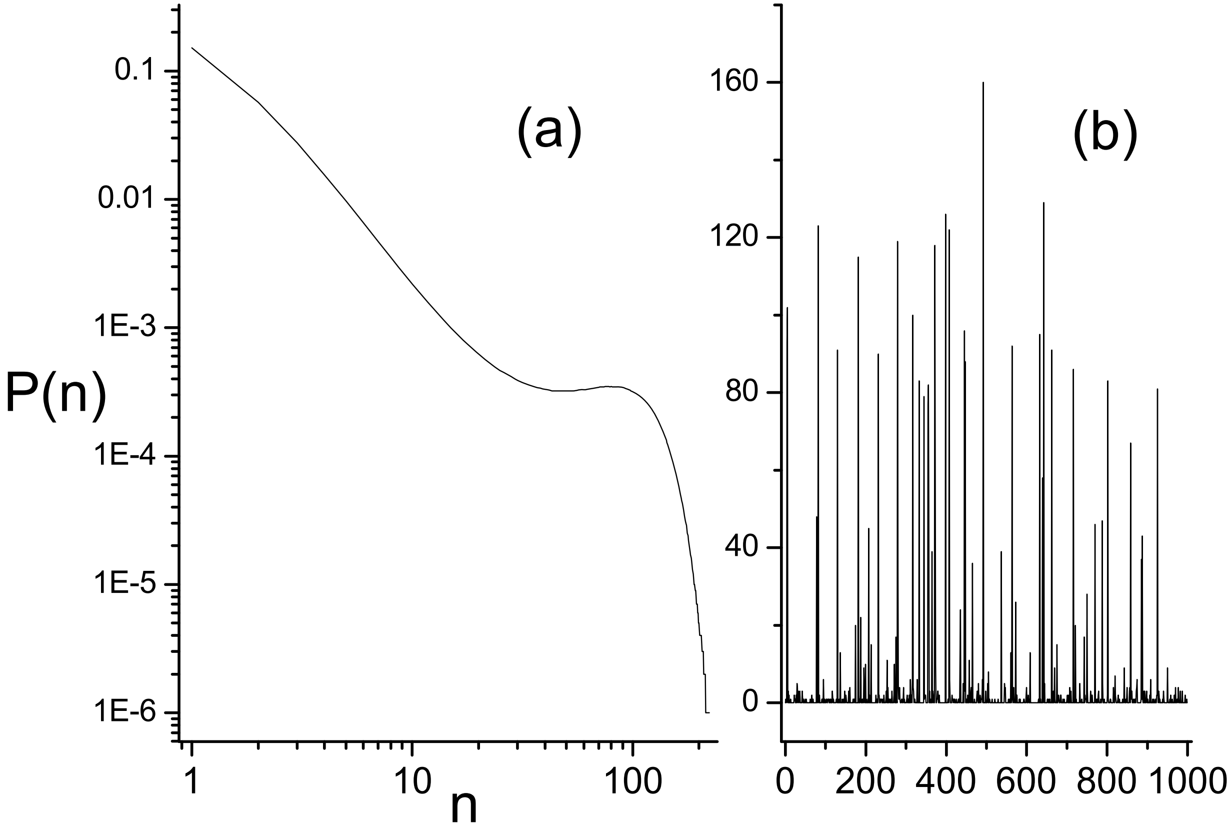


Typical occupation configuration obtained from simulation (b=3):





L=1000, N=2000, b=3



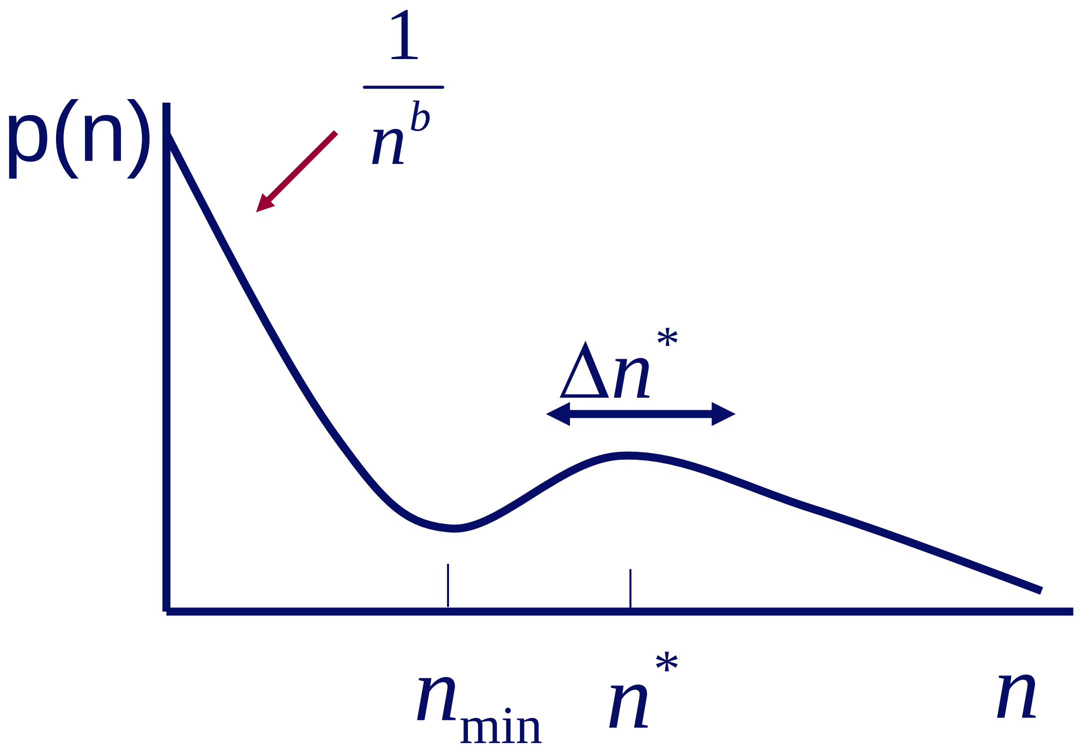
L=1000 b=4 k=1 $\rho = 4$

Analysis of the occupation distribution

$$p(n) \propto \prod_{k=1}^{n} \frac{z}{u(k)} \qquad u(n) = 1 + \frac{b}{n} + c\left(\frac{n}{L}\right)^{k}$$

$$p(n) \propto \frac{z^{n}}{n^{b}} \exp\left[-a\frac{n^{k+1}}{L^{k}}\right] \qquad a = c/(k+1)$$

the fugacity z (L) is determined by the density: $\sum_{n} np(n) = N / L$



$\frac{1}{n^b}$
p(n)
Δn^*
$n_{\min}$
n^*
n

Up to logarithmic corrections the peak parameters scale with the system size as:

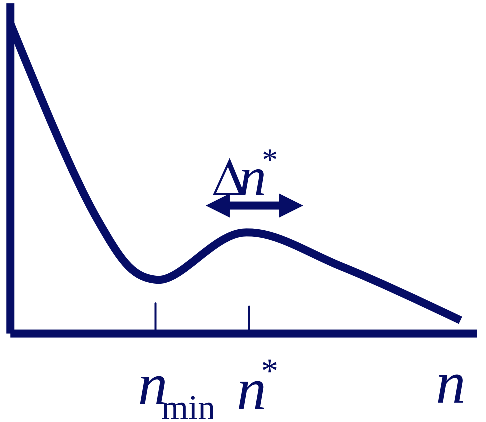
$$n^* \propto L^{k/(k+1)}$$

$$\Delta n^* \propto L^{k/(k+1)}$$

$$p(n^*) \propto L^{-2k/(k+1)}$$

$$w = p(n^*)\Delta n^* \propto L^{-k/(k+1)}$$

("condensate" weight)



The peak is broad

Number of particles in a condensate: $n^* \propto L^{k/(k+1)}$

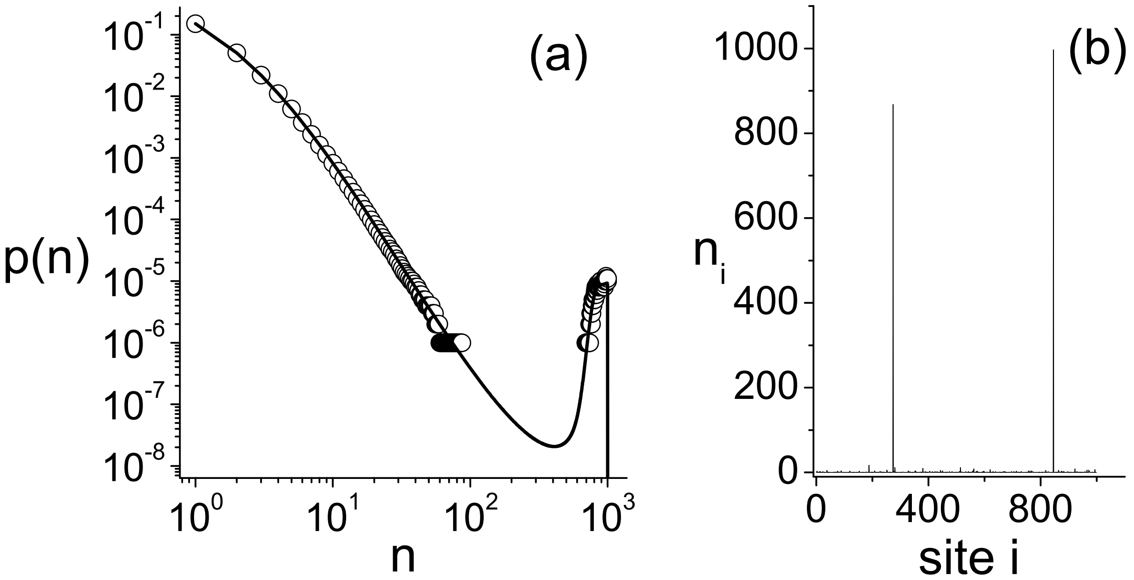
Number of condensates: $Lw = Lp(n^*)\Delta n^* \propto L^{1/(k+1)}$

The condensed phase is composed of a large number (sub-extensive) of meso-condensates each contains a sub-extensive number of particles such that the total occupation of all meso-condensates is extensive.

Another choice of u(n) – a sharp (exponential) cutoff at large densities:

$$u(n) = 1 + \frac{b}{n} + \exp(n - aL)$$

Here we expect condensates to contain up to aL particles (extensive occupation), and hence a finite number of condensates.



L=1000, N= 2300, a=1, b=4
critical density = 0.5

Results of scaling analysis:

$$n^* \approx aL - \ln L$$

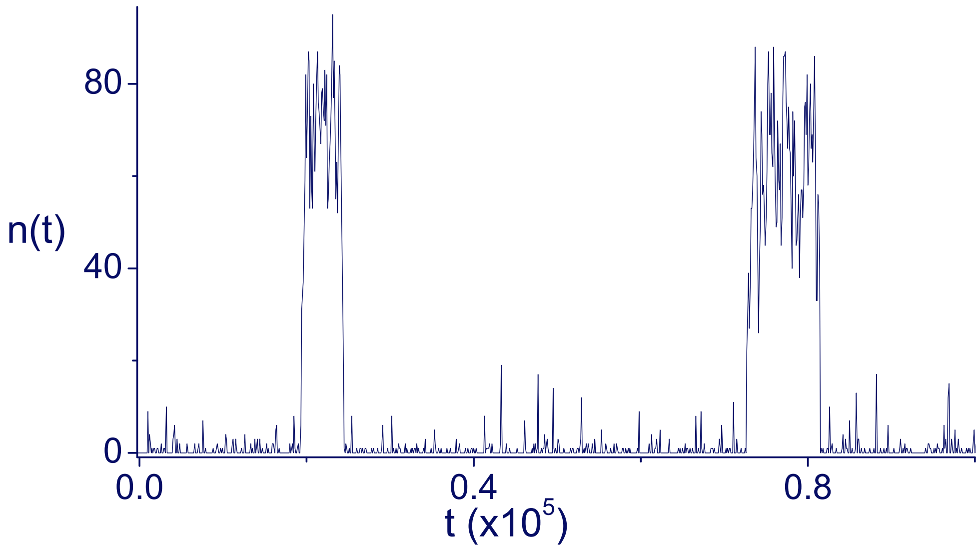
$$\Delta n^* \propto L^{1/2}$$

$$p(n^*) \propto L^{-3/2}$$

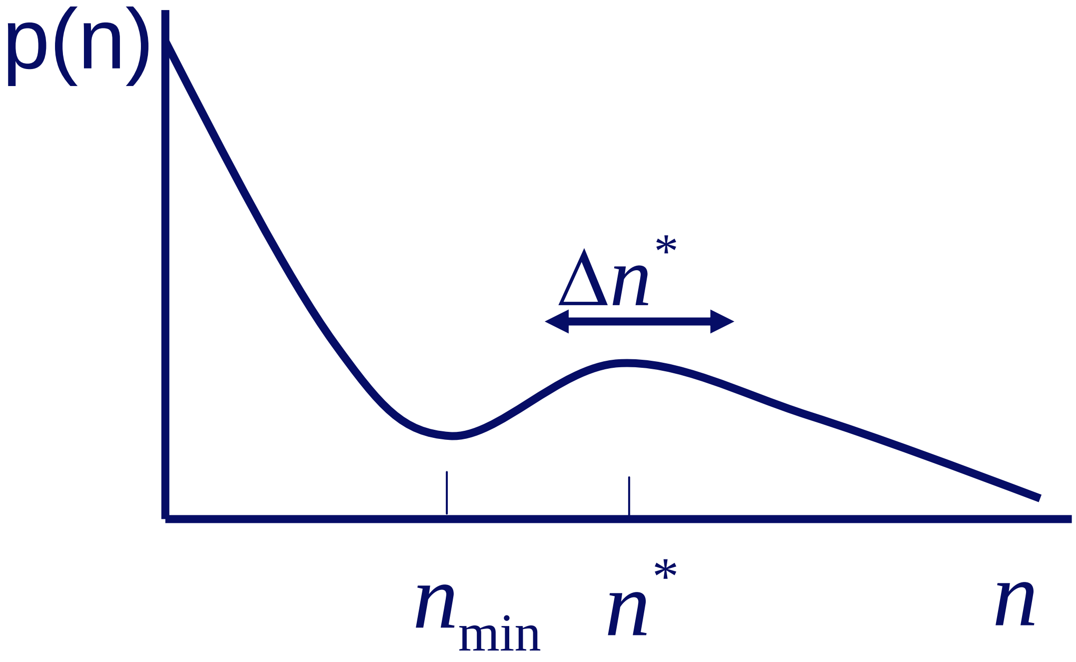
$$w = p(n^*)\Delta n^* \propto 1/L$$

number of condensates is Lw = O(1)

Dynamics



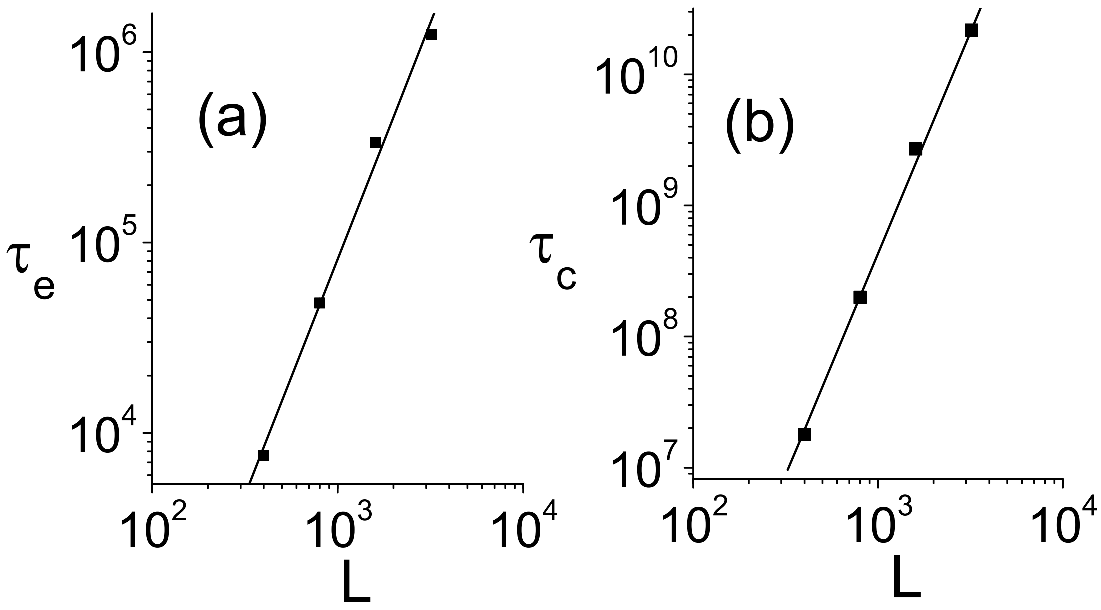
temporal evolution of the occupation of a single site



Arrhenius law approach:

creation time $\tau_c \propto \dfrac{1}{p(n_{\min})} \propto L^{bk/(k+1)}$

evaporation time $\tau_e \propto \dfrac{w}{p(n_{\min})} \propto L^{(b-1)k/(k+1)}$



b=4, k=5, density=3

Can one have a critical phase for a whole range of densities (like self organized criticality)?

- A ZRP model with non-conserving processes
- Connection to network dynamics

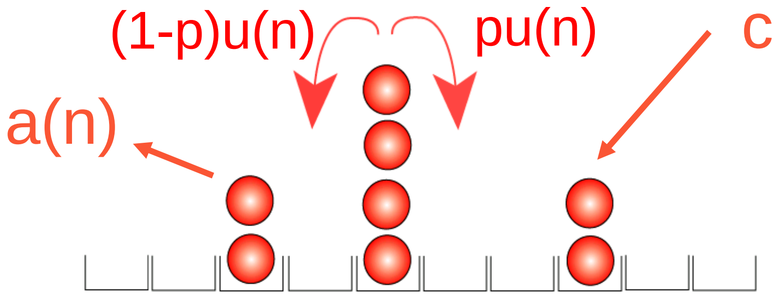
Non conserving ZRP

A. Angel, M.R. Evans, E. Levine, D. Mukamel, PRE 72, 046132 (2005);
JSTAT P08017 (2007)

hopping rate: $$u(n) = 1 + \frac{b}{n}$$

creation rate: $$c = \frac{1}{L^s}$$

annihilation rate: $$a(n) = \left(\frac{n}{L}\right)^k$$



Evolution equation (fully connected)

$$\frac{\partial p(n)}{\partial t} = \left[u(n+1) + a(n+1)\right]p(n+1) - (\lambda + c)\,p(n)$$
$$-\left[u(n) + a(n)\right]p(n) + (\lambda + c)\,p(n-1)$$

hopping current $\lambda = \sum_{n=1}^{\infty} u(n)\,p(n)$

Sum rules:

normalization
$$\sum_{n=0}^{\infty} p(n) = 1$$

hopping current
$$\lambda = \sum_{n=1}^{\infty} u(n)\, p(n)$$

steady state density
$$\sum_{n=1}^{\infty} a(n)\, p(n) = c$$

Steady state distribution

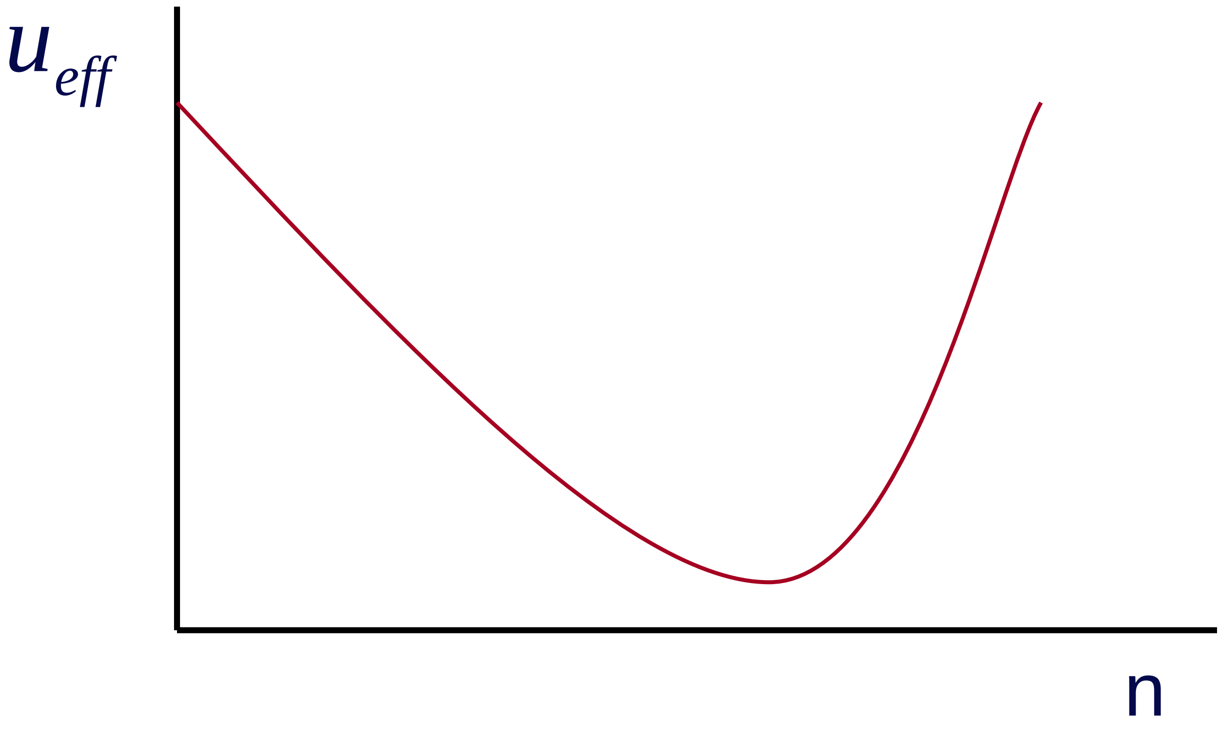
$$p(n) = \frac{(\lambda + c)^n}{\prod_{m=1}^{n}\left[a(m) + u(m)\right]} p(0)$$

Like a conserving ZRP with an effective hopping rate

$$u_{eff}(n) = u(n) + a(n) = 1 + \frac{b}{n} + \left(\frac{n}{L}\right)^k$$

and $z = \lambda + c$

$$u_{eff}(n) = 1 + \frac{b}{n} + \left(\frac{n}{L}\right)^{k}$$



Steady state distribution

$$p(n) = \frac{(\lambda + c)^n}{\prod_{m=1}^{n}\left[a(m) + u(m)\right]} p(0)$$

λ Is determined by the sum rule $c = \sum_{n=1}^{\infty} a(n) p(n)$

$$L^{-s} = \sum_{n=1}^{\infty} \left(\frac{n}{L}\right)^k p(n)$$

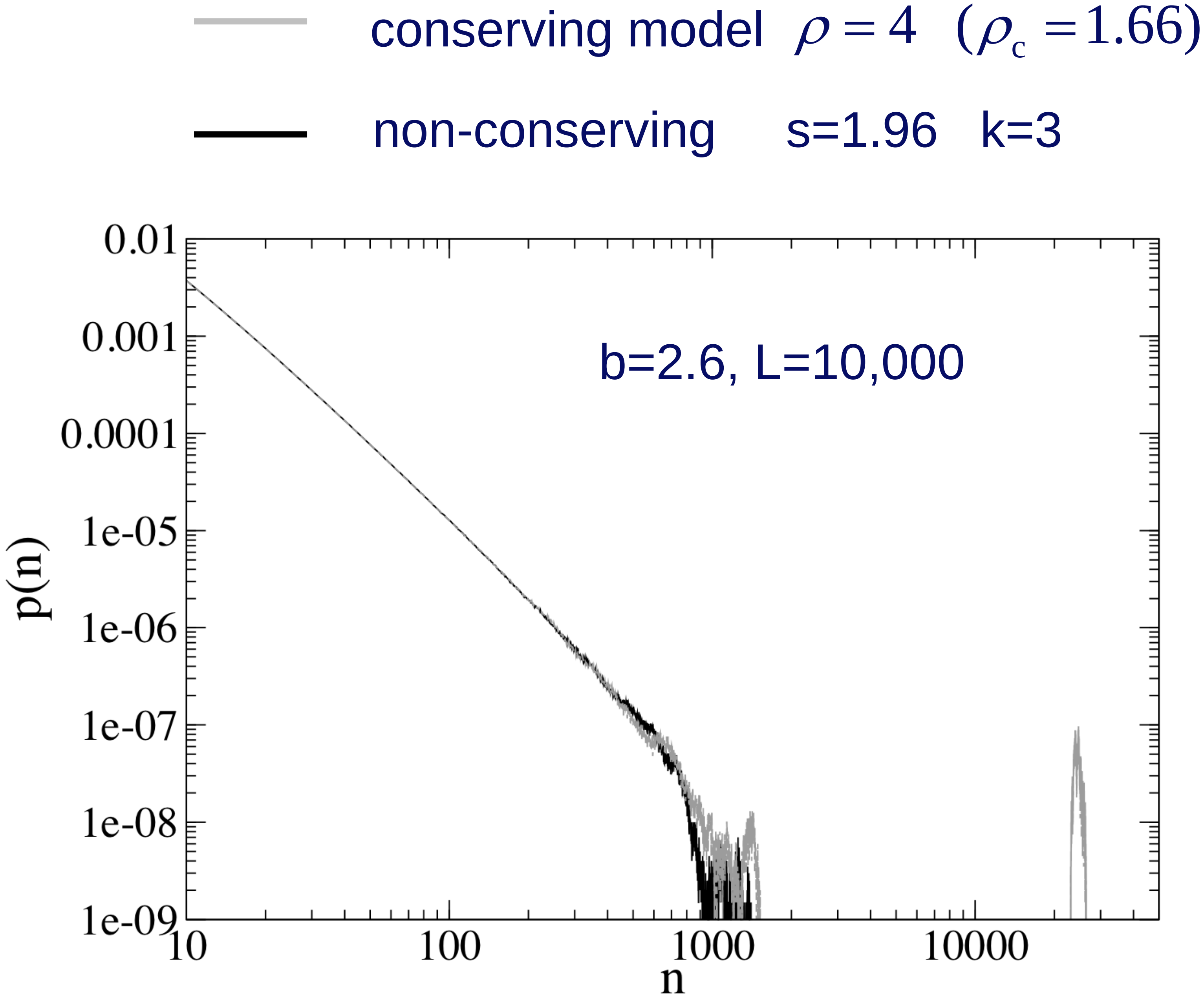
$$L^{k-s} = \sum_{n=1}^{\infty} n^k p(n)$$

(s,k) phase diagram

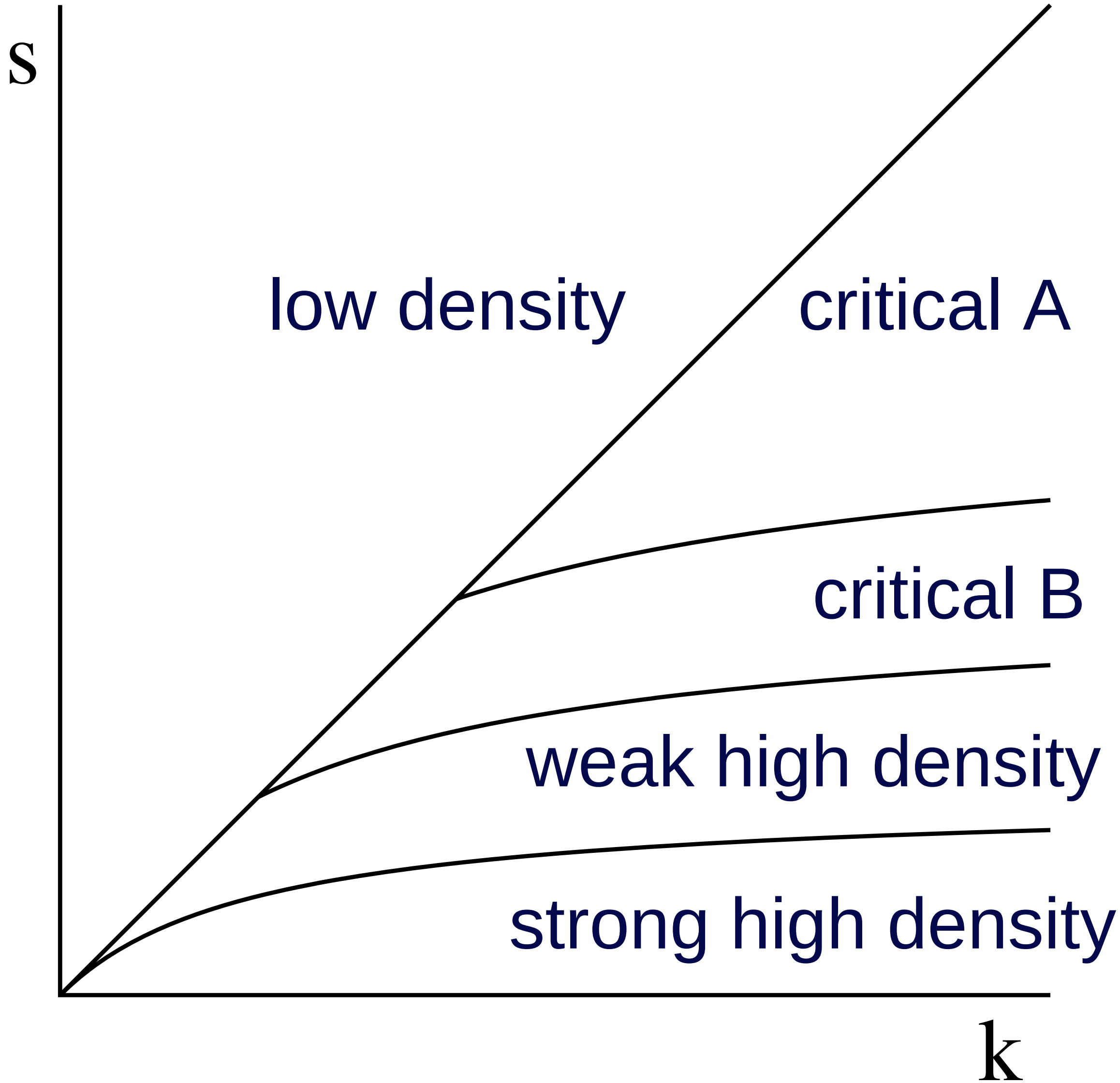
large s - small creation rate ($1/L^s$) - low density

small s – large creation rate – high density

intermediate s ?


conserving model $\rho = 4$ ($\rho_c = 1.66$)
non-conserving s=1.96 k=3
b=2.6, L=10,000
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
p(n)
10
100
1000
10000
n

Phase diagram



$$p(n) = \frac{(\lambda + c)^n}{\prod_{m=1}^{n}\left[a(m) + u(m)\right]} p(0)$$

$$u(n) + a(n) = 1 + \frac{b}{n} + \left(\frac{n}{L}\right)^k$$

$$P(n) = \frac{1}{n^b} e^{-\frac{n^{k+1}}{(k+1)L^k}} (\lambda + c)^n$$

$$L^{k-s} = \sum_{n=1}^{\infty} n^k p(n)$$

Low density phase $s > k$

$$p(n) \approx L^{-(s-k)n}$$

$$\rho = p(1) \approx L^{k-s}$$

The density vanishes in the thermodynamic limit.
The lattice is basically empty.

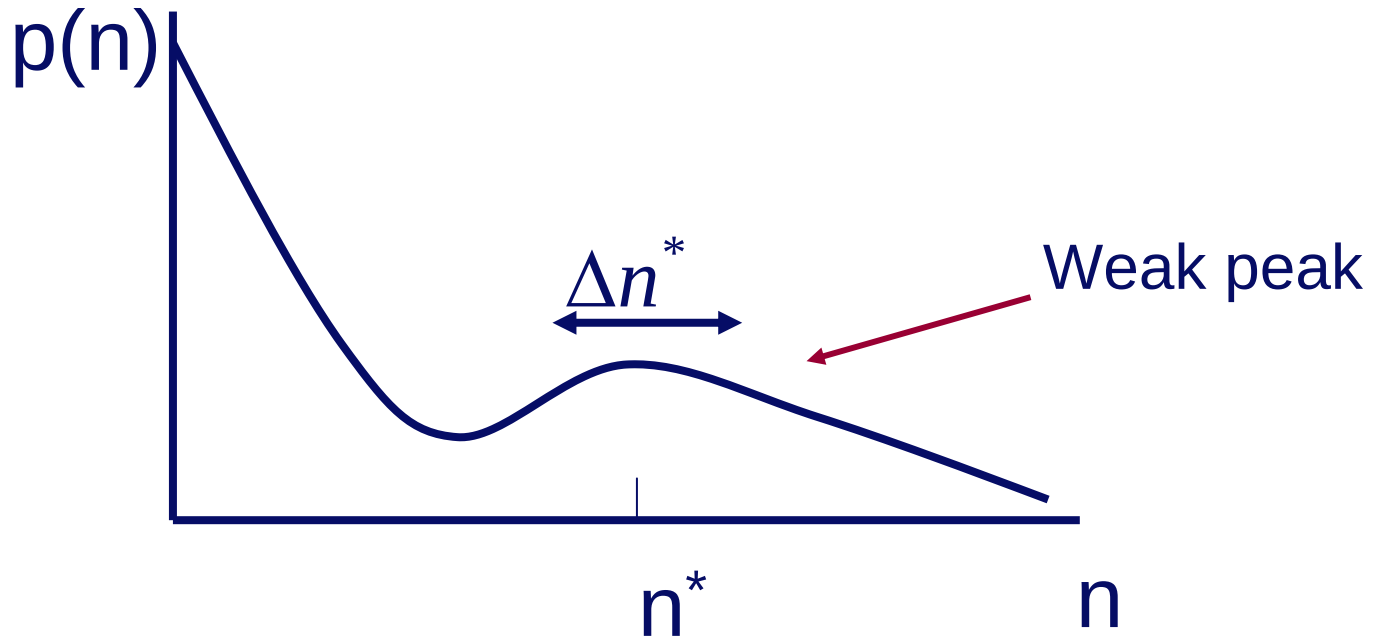
Critical phase A $bk/(k+1)<s<k$

$$P(n) \approx \frac{1}{n^b} e^{-g\frac{n}{L^y}}$$

$$y = \frac{k-s}{k-b+1}$$

critical phase with a cutoff at L^y which diverges in the thermodynamic limit

Critical phase B   2k/(k+1)<s<bk/(k+1)



$$n^* \approx \Delta n^* \approx L^{k/(k+1)}$$

$$p(n^*) \approx L^{-s}$$

$$w = p(n^*)\Delta n^* \approx L^{k/(k+1)-s}$$

$$n_{wp} \approx n^* w \approx L^{2k/(k+1)-s} \to 0$$

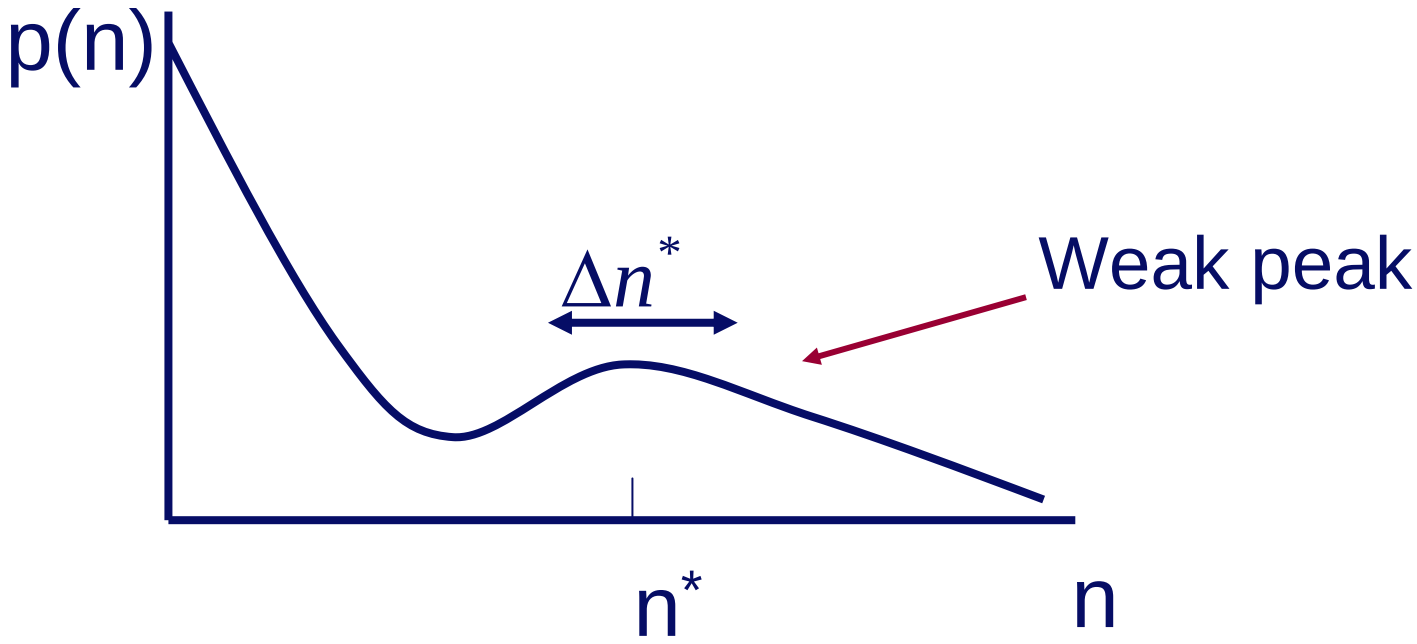
$$\rho \to \rho_c$$

Most of the sites form a fluid with typically low occupation. In addition a sub-extensive number of sites Lw are highly occupied with $n=n^*$ (meso-condensates)

no. of condensates $Lw \approx L^{1-s+k/(k+1)}$

condensate occupation $n^* \approx L^{k/(k+1)}$

Weak high density $k/(k+1)<s<2k/(k+1)$



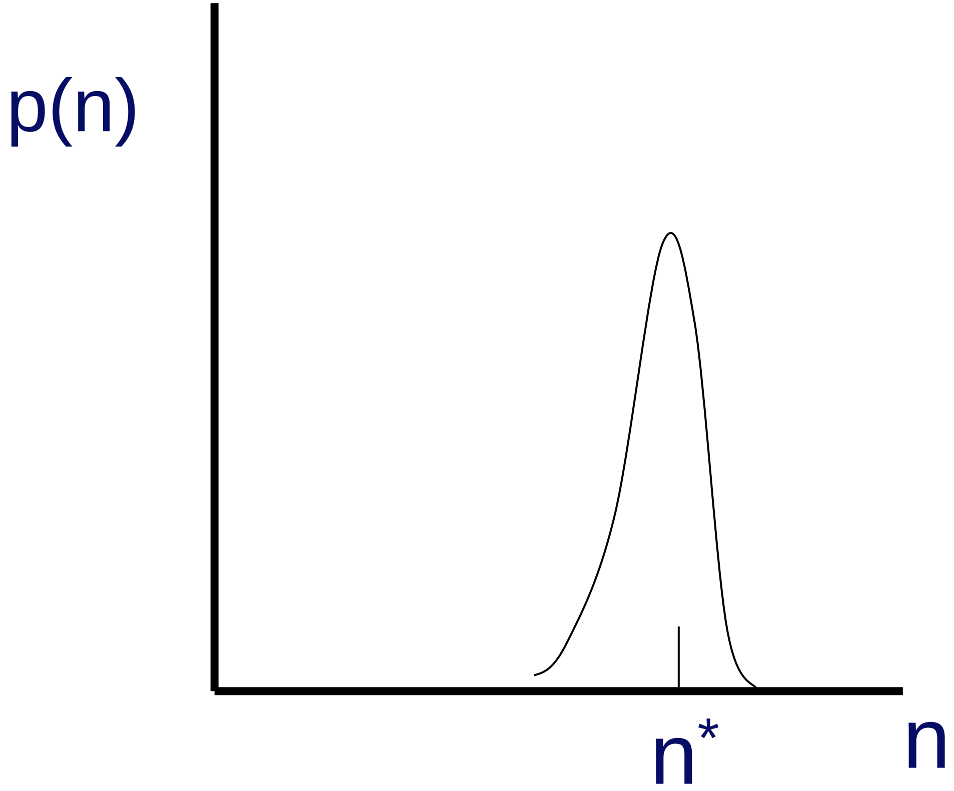
$$n^* \approx \Delta n^* \approx L^{k/(k+1)}$$

$$p(n^*) \approx L^{-s}$$

$$w = p(n^*)\Delta n^* \approx L^{k/(k+1)-s}$$

$$n_{wp} \approx n^* w \approx L^{2k/(k+1)-s} \to \infty$$

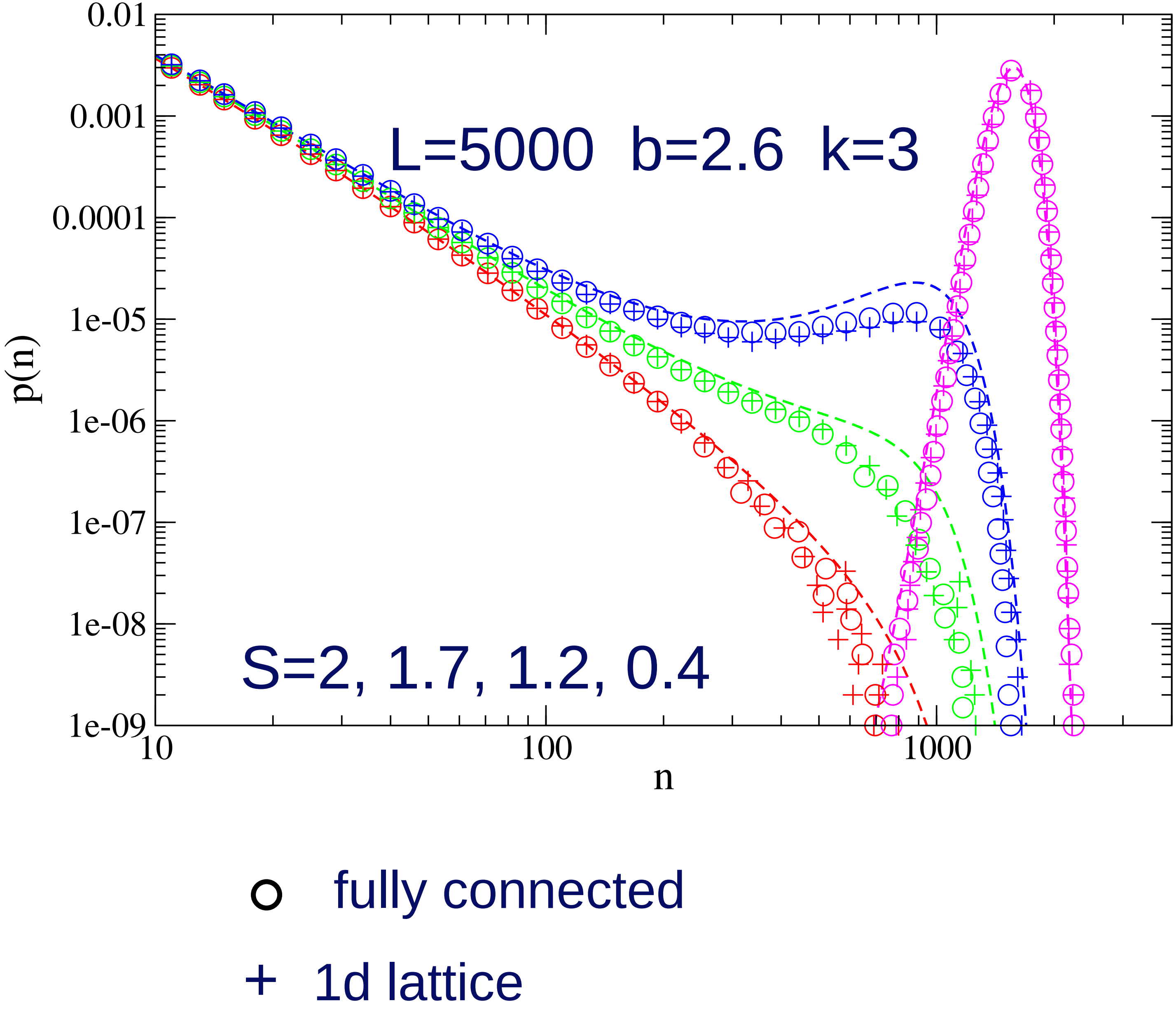
Strong high density phase   s<k/(k+1)



$$n^* \approx L^{1-s/k}$$

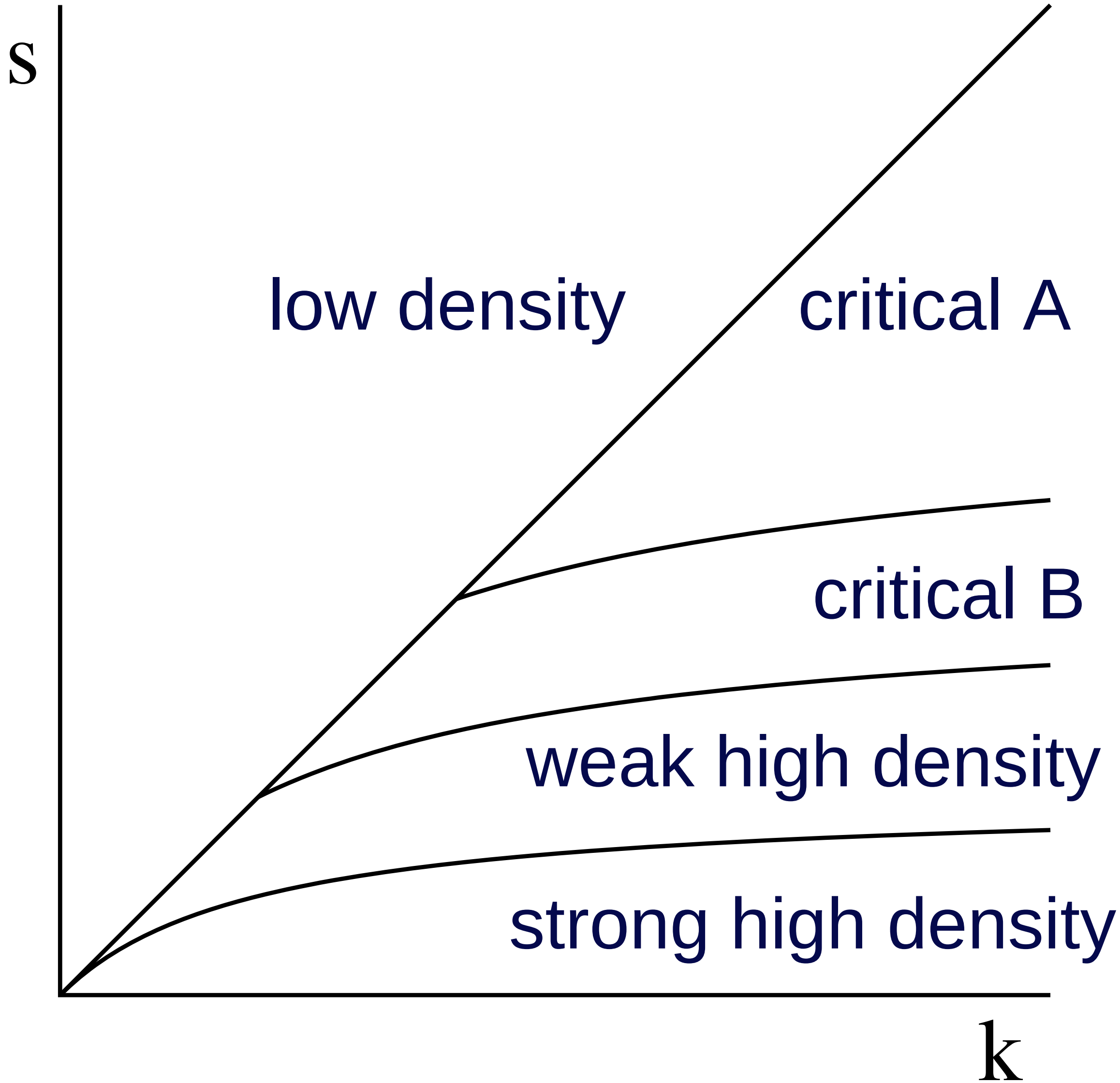
$$w = 1$$

no fluid, all site are highly occupied by n^* particles



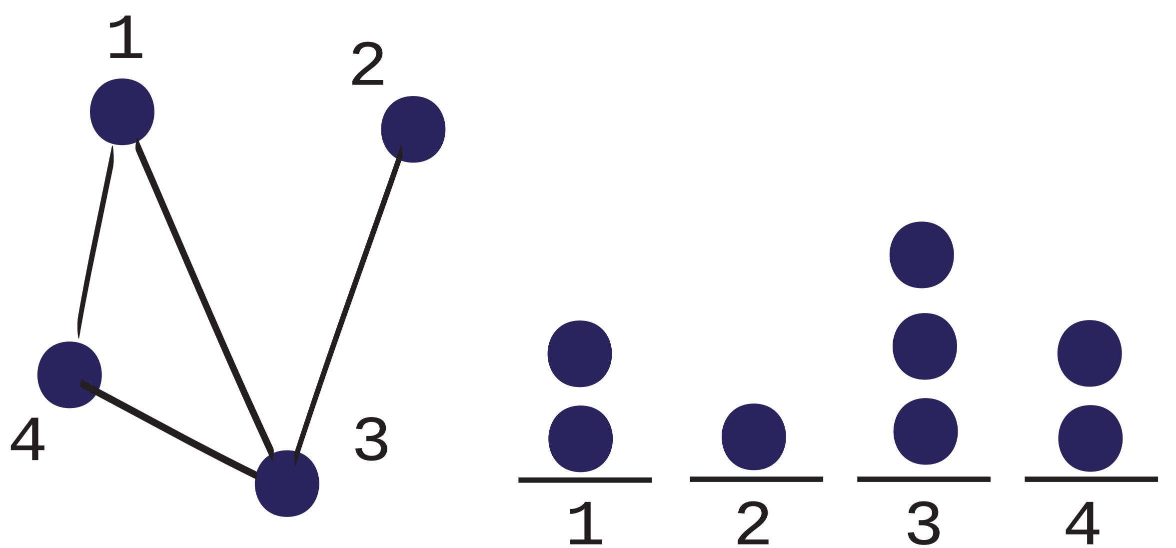
L=5000 b=2.6 k=3
S=2, 1.7, 1.2, 0.4
p(n)
n
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
10
100
1000
fully connected
1d lattice

Phase diagram



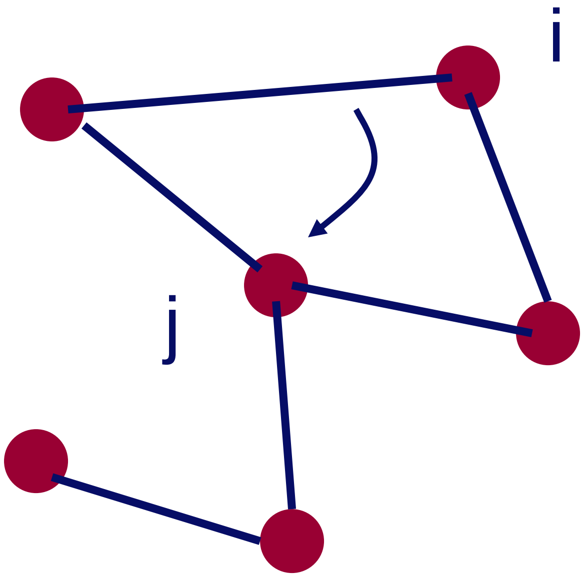
Network dynamics

correspondence between networks and ZRP



node --- site   link to a node --- particle

Rewiring dynamics

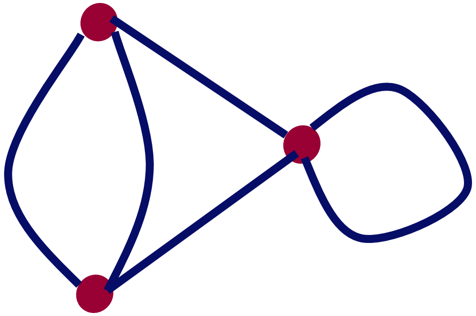


$$n_i \rightarrow n_i - 1$$

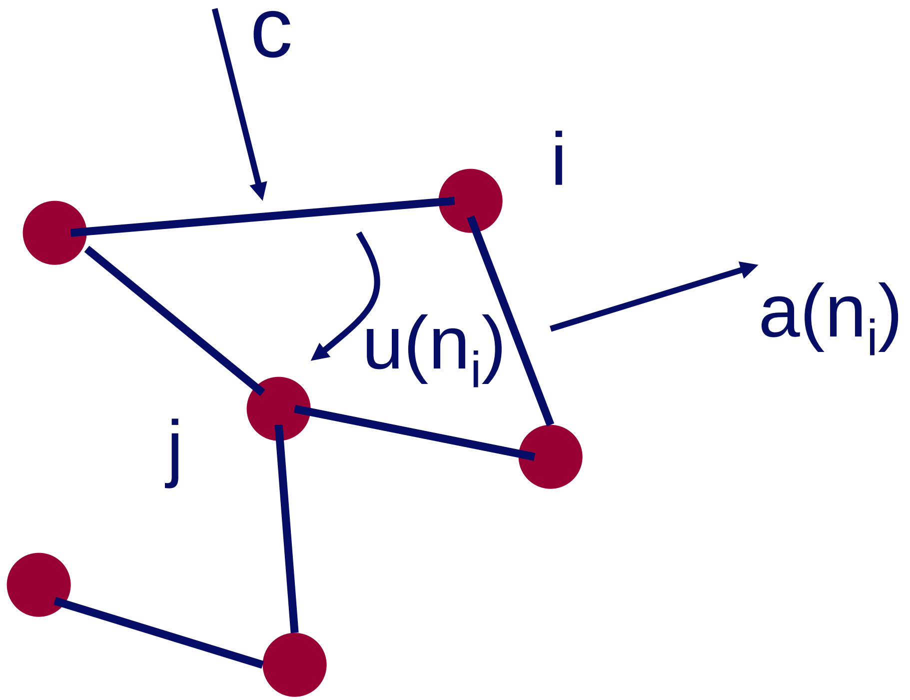
$$n_j \rightarrow n_j + 1$$

rate: $u(n_i)$

Networks with multiple and self links (tadpoles).



rewiring – creation – evaporation processes



one obtains the same results as for the non-conserving ZRP

$$\frac{\partial p(n)}{\partial t} = \left[u(n+1) + a(n+1) + \Lambda(n+1)\right]p(n+1)$$
$$-(\lambda + 2c)\,p(n) - \left[u(n) + a(n) + \Lambda(n)\right]p(n)$$
$$+(\lambda + 2c)\,p(n-1)$$

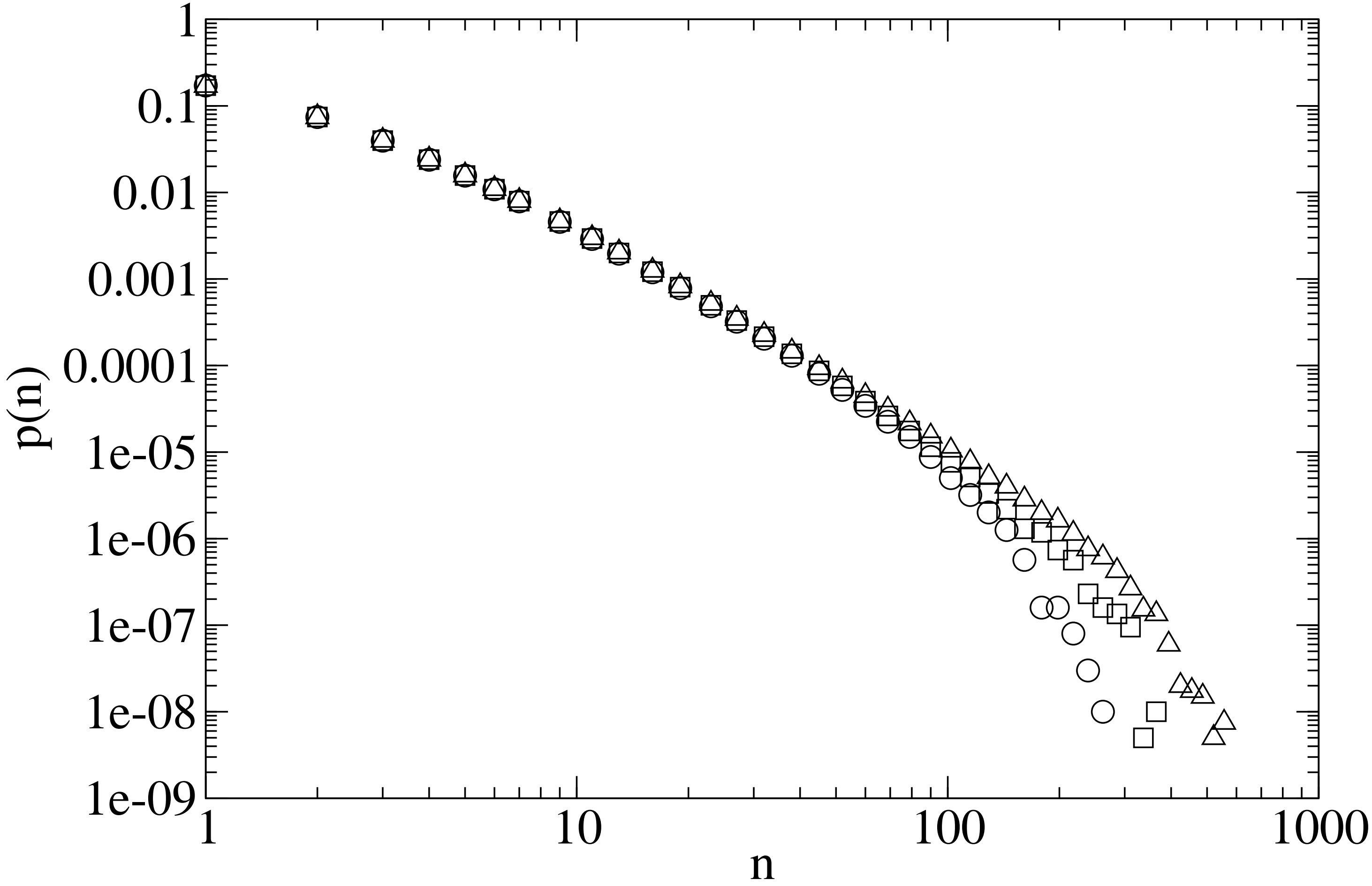
Here $\Lambda(n)$ is the evaporation current resulting from other nodes

$$\Lambda(n) = \frac{n}{N}\sum_{l=1}^{\infty} a(l)\,p(l) = c\frac{n}{N}$$

$$\lambda = \sum_{l=1}^{\infty} u(l)\,p(l)$$

Hence

$$u_{eff} = u(n) + a(n) + \Lambda(n) = 1 + \frac{b}{n} + \left(\frac{n}{L}\right)^{k} + c\,\frac{n}{N}$$



b=2.6 k=3 s=2 L=1000, 2000, 4000

summary

- The Zero-Range-Process may be used to probe ordering phenomena in driven systems.
- Multiple meso-condensates may result in certain cases
- Generic critical phase (like self organized criticality)
- Network dynamics may exhibit similar phenomena