Diffusion for coupled maps with a conservation law

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joint work with J. Bricmont

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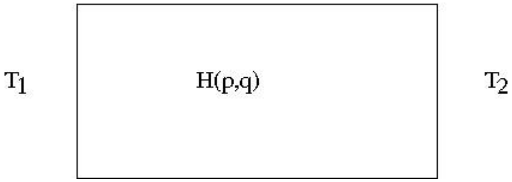
Heat conduction

- **Nonequlibrium stationary state**: finite system, heat bath (noise) on the boundary



- **Approach to equilibrium**: Infinite system, initial state with constant temperature at spatial infinity.
- Models: **Coupled maps** and **Coupled flows**
  - Subsystems (maps or flows) indexed by $x \in \mathbb{Z}^d$
  - Couple together locally: system at x interacts with systems at nearby y.

Local energy and flux

Total energy is sum of **local energies** H_x:

$$H = \sum_x H_x$$

No coupling: each H_x is conserved, $\dot{H}_x = 0$.

Turn on coupling: only H is conserved and

$$\dot{H}_x = -\nabla \cdot J_x$$

J_x **flux of energy** at site x.

Show: H_x **diffuses** and J_x is tied to to H_x by **Fourier's law**.

Return to equilibrium

Infinite system $V = \mathbb{Z}^d$, $\mu_t = \phi_t \mu_0$, ϕ_t Hamiltonian flow

Diffusion in **hydrodynamic limit**:

- Take μ_0 s.t. $E_{\mu_0} H_x = \tau(\epsilon x)$
- Let

$$\tau(t, x) := \lim_{\epsilon \to 0} E_{\mu_{t/\epsilon^2}} H_{x/\epsilon}$$

$$j(t, x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_{\mu_{t/\epsilon^2}} J_{x/\epsilon}$$

- Show:

$$j = -\kappa(\tau)\nabla\tau \quad \text{Fourier law}$$

$$\dot{\tau} = \nabla \cdot (\kappa(\tau)\nabla\tau) \quad \text{Diffusion}$$

NESS

Fix $\Lambda \subset \mathbb{R}^d$ and $t : \partial\Lambda \to \mathbb{R}^+$.

Lattice box $V = \Lambda/\epsilon \cap \mathbb{Z}^d$,

- Hamiltonian dynamics in V
- noise of energy $T_x = t_{x/\epsilon}$ at $x \in \partial V$.

For $x \in \Lambda$ let

$$j_x = \frac{1}{\epsilon} E\, J_{x/\epsilon} \quad \tau_x = E\, H_{x/\epsilon}$$

E expectation in NESS

Then, as $\epsilon \to 0$ show

- **Fourier law**: $j(x) = -\kappa(\tau(x))\nabla\tau(x)$
- **Temperature profile**:

$$\nabla \cdot (\kappa(\tau)\nabla\tau) = 0$$
$$\tau|_{\partial\Lambda} = t$$

Coupled dynamics

Models: **Coupled flows** and **Coupled maps**

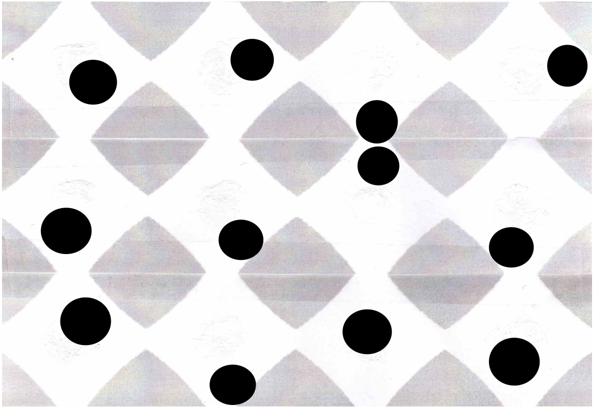
1. **Weakly anharmonic coupled oscillators**:
   - At each $x \in \mathbb{Z}^d$ (an)harmonic oscillator with (an)harmonic interaction with nearest neighbours
   - Lots of numerics.
   - In a weak anharmonicity scaling limit (**kinetic limit**) get formally a Boltzman equation (Spohn)
   - Diffusion and Fourier proved there (J.B., A.K. math-ph 0703014)
   - Hard to prove kinetic limit

2. **Weakly coupled chaotic systems**
   - Coupled billiards or Anosov systems
   - Coupled maps with a local conservation law

Coupled chaotic systems

Weakly coupled strongly chaotic flows

- Bunimovich, Liverani, Pellegrinotti, Suhov, Eckmann, Young, Gaspard, Gilbert,....



Coupled map lattice

Coupled map lattice: phase space $\mathcal{M} = M^{\mathbb{Z}^d}$.

Local dynamics: $\phi : M \to M$

Uncoupled dynamics $\Phi : \mathcal{M} \to \mathcal{M}$

$$\Phi(m)_x = \phi(m_x)$$

Coupling map $\Psi : \mathcal{M} \to \mathcal{M}$

Ψ **local**: $\Psi(m)_x$ depends weakly on m_y for $|y - x|$ large.

CML dynamics: $\Psi \circ \Phi : \mathcal{M} \to \mathcal{M}$

Coupled map lattice with a conservation law

Dolgopyat, Liverani.....

We take $M = \mathbb{R}_+ \times N$ and

$$\phi(E, \theta) = (E, f(\theta)), \quad E \in \mathbb{R}_+, \ \theta \in N$$

with f hyperbolic, e.g. $N = S^1$, $f(\theta) = 2\theta$, or $N = \mathbb{T}^2$, f torus automorphism

- Energy of each cell is conserved: $E_x \to E_x$ i.e. one vanishing Lyapunov exponent per unit volume.
- Chaotic dynamics for the rest: $\theta_x \to f(\theta_x)$
- Coupling typically removes the degeneracy
- Look for coupling so that **total energy** $E = \sum_x E_x$ is conserved.

Coupling

Coupling: nearby cells interact, exchange energy

$$E'_y = \sum_{|x-y|=1} p_{xy}(E,\theta)E_x$$

$$\theta'_x = f(\theta_x) + g_x(E,\theta)$$

- p_{xy}, g_x depend on θ_u, E_u for u near x only
- $p_{xy} \geq 0$
- $\sum_y p_{xy}(E,\theta) = 1$ for all E, θ

Total energy $\sum_x E_x$ conserved.

Diffusion

Let, at $t = 0$, $E_x \to T$ as $|x| \to \infty$.
Show $E_x(t)$ diffuses to T **almost surely** in $\theta(0)$

$$E_x(t) - T \sim t^{-d/2} f(x/\sqrt{t})$$

Hydrodynamic scaling limit:

- Let $E_x(0) = \tau(\epsilon x)$
- Show: $\lim_{\epsilon \to 0} E(t/\epsilon^2, x/\epsilon) = \tau(t, x)$ satisfies

$$\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau)$$

almost surely in $\theta(0)$.

Random walk

Iteration of coupled maps

$$E_y(t+1) = \sum_{|x-y|=1} p_{xy}(E(t),\theta(t))E_x$$

$$\theta_x(t+1) = f(\theta_x(t)) + g_x(E(t),\theta(t))$$

where $p_{xy} \geq 0$ and

$$\sum_y p_{xy}(E,\theta) = 1.$$

- $p_{xy}(E(t),\theta(t)) := p_{xy}(t)$ can be viewed as **transition probabilities** of a random walk
- $E_x(t)$ is (proportional to) the probability of finding the walker at x at time t

Random walk in random environment

- Transition probabilities $p_{xy}(t)$ depend on space and time: random walk in a **space-time dependent environment**
- $p_{xy}(t)$ completely determined by initial conditions of E, θ
- Prove: walk is diffusive **almost surely** in $\theta|_{t=0}$
- Typical $\theta|_{t=0} \implies$ **random** $p_{xy}(t)$
- Prove **quenched** CLT for such walks i.e. a.s. in the p-ensemble

What is the statistics of p like?

Space time mixing environment

Suppose first p_{xy} and g_x depend **only** on θ i.e.

$$
\begin{aligned}
E_y(t+1) &= \sum_{|x-y|=1} p_{xy}(\theta(t))E_x \\
\theta_x(t+1) &= f(\theta_x(t)) + g_x(\theta(t))
\end{aligned}
$$

Then:

- f hyperbolic, , g small, smooth $\Longrightarrow$ $f+g$ hyperbolic CML $\Longrightarrow$ θ-dynamics space time mixing
- $\Longrightarrow$ $p_{xy}(t)$ **weakly correlated** in space and time
- Use **Renormalization** to prove randomness is **irrelevant** $\Longrightarrow$
- Random walk satisfies CLT a.s. in p $\Longrightarrow$
- E diffuses almost surely in $\theta|_{t=0}$ (arxiv 05/08).

Slow and fast variables

Suppose p_{xy} depends on E too

- θ still space time mixing
- $p(\theta(t), E(t))$ gets **long range correlations** through E dependence
- RG $\implies$ E dependence **irrelevant** $\implies$
- CLT still holds (in preparation...)

Suppose also g_x depends on E

- **Fast** variables θ get slaved to the slow ones E

Deterministic diffusion is reduced to the study of RW in weakly correlated environment.

Diffusion

Random walk with space time dependent transition probability $p_{xy}(t)$.

Probability of a walk $\omega = (\omega_0, \dots, \omega_T)$ in time T

$$P^T(\omega) = \prod_{t=0}^{T-1} p_{\omega_t \omega_{t+1}}(t).$$

E_T expectation in walks with $\omega_0 = 0$. Diffusion constant

$$D_T = T^{-1} E_T \omega(T)^2$$

Diffusion:

$$\lim_{T \to \infty} D_T = D$$

Scaling limit

Rescale to space Ω of paths $\omega : [0,1] \to \mathbb{R}^d$

$$\omega(t) = T^{-\frac{1}{2}} \omega_{Tt}$$

E_T induces expectation $\mathcal{E}_T$ on such paths

$$\mathcal{E}_T F(\omega(\cdot)) = E_T F(T^{-\frac{1}{2}} \omega_{T\cdot})$$

Scaling limit

$$\lim_{T\to\infty} \mathcal{E}_T F := \mathcal{E} F$$

for $F : \Omega \to \mathbb{R}$ continous on path space.
Prove: **allmost surely** in the p ensemble $\mathcal{E}$ exists and equals Wiener measure, diffusion constant D:

$$D = \lim_{T\to\infty} D_T = \mathcal{E}\omega(1)^2$$

Renormalization group

Probability to walk from x to y during time interval $[t, t']$:

$$P_{t,t'}(x, y, p) = (p(t) \dots p(t' - 1))_{xy}$$

Define **renormalized transition probability** matrix

$$(R_l p)_{xy}(t) = l^d P_{l^2 t, l^2 (t+1)}(lx, ly, p)$$

for walks on $l^{-1}\mathbb{Z}^d$. Then, if l^2 divides t, t',

$$P_{t,t'}(x, y, p) = l^{-d} P_{t/l^2, t'/l^2}(l^{-1}x, l^{-1}y, R_l p).$$

$R_l p$ is the **Renormalization group** flow in a space of random matrices.

Asymptotics

Scaling limit controlled by R_l as $l \to \infty$

- Diffusion constant at time t:

$$D(t,p) = t^{-1} \sum_x P_{0,t}(0,x,p)x^2$$

reduces to **unit time** one with rates $R_l p$:

$$D(l^2,p) = D(1,R_l p).$$

- Let $F : \Omega \to \mathbb{R}^d$ depend on ω restricted to $\tau^{-1}\mathbb{Z}$ and $l^2 = T/\tau$. Then

$$\mathcal{E}_T F(\omega(\cdot)) = E_\tau^{R_l p} F(\tau^{-\frac{1}{2}} \omega_{T\cdot})$$

$=$ **fixed time** τ problem with rates $R_l p$.

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Fixed point

If $\mathrm{R}_l p \to p^*$ as $l \to \infty$ then

$$D = D(1, p^*)$$

and scaling limit is given by

$$\mathcal{E} F(\omega(\cdot)) = E_{\tau}^{p^*} F(\tau^{-\frac{1}{2}} \omega_{\tau\cdot}).$$

Convergence to Wiener measure:

$$p^*_{xy} = (2\pi D)^{-d/2} e^{-\frac{(x-y)^2}{2D}}.$$

Thus, we want to prove $R_l p$ becomes **nonrandom** as $l \to \infty$ and converges to p^* a.s. in p.

Semigroup

R_l satisfies $R_{ll'} = R_l R_{l'}$.

Study R_l **iteratively**:

- Pick $L > 1$ and let $R := R_L$
- Let $p_n = R^n p$ i.e. $p_n = R_{L^n} p$
- Let E be expectation in p ensemble. Write

$$p_n = Ep_n + b_n.$$

- Show
  - $b_n \to 0$ **almost surely**
  - $Ep_n \to p^*$.

Assumptions

Assume

- Distribution of p translationally and rotationally invariant
- $Ep_{xy} = T(x - y)$ exponentially decaying
- Cumulants of p cluster exponentially

$$E(p_{x_1 y_1}(t_1); p_{x_2 y_2}(t_2); \dots; p_{x_N y_N}(t_N)) \leq \epsilon^N e^{-\lambda\tau},$$

τ length of shortest tree on the space time support

Assumptions are satisfied by

- $p(\theta)$ analytic, local with θ analytic CML
- $p(s)$ local in spins of a high temeperature Ising model and the like

Result

Theorem: In all dimensions $d \geq 1$

$$E(R^n p_{x_1 y_1}(t_1); R^n p_{x_2 y_2}(t_2); \ldots; R^n p_{x_N y_N}(t_N)) \leq \epsilon_n^N e^{-\lambda\tau},$$

with $\epsilon_n \to 0$ as $n \to \infty$ (exponentially).

Randomness **irrelevant** in all dimensions

Implications: D exists, scaling limit Wiener

General case

Include E dependence of the walk

$$E_y(t+1) = \sum_{|x-y|=1} p(\theta(t), E(t))_{xy} E_x(t)$$

Environment depends on the trajectory $E(t)$.

$p(s)$, $p(t)$ **diffusively** correlated.

RG for **conditional** transition probabilities.

- Let $E_n(t,x) = L^{nd} E(L^{2n}t, L^n x)$
- $p_n(t) = p_n(t, E_n(t))$ is conditioned on $E_n(t)$ i.e. collects rescaled walks on time interval $[L^{2n}t, L^{2n}(t+1)]$ conditioned on $E(L^{2n}t)$.
- $\{p_n(t,E)\}$, E fixed **exponentially** weakly correlated
- E dependence an **irrelevant** perturbation in the RG.

Analogy in continuum

Our CML is a discrete version of the SPDE

$$\dot{E} = \partial_\mu (a_{\mu\nu}(E)\partial_\nu E + b_\mu(E)E)$$

where $a_{\mu\nu}$ and b_μ are random and nonlinear

The RG produces a non random PDE in the scaling limit

$$\dot{E} = \partial_\mu (\kappa_{\mu\nu}(E)\partial_\nu E)$$

It is a combination of RG for PDE's (J.B. & A.K., 1992) and RG for RWRE (J.B. & A.K., 1991).

However, the randomness is deterministic

Hamiltonian systems

What kind of CML should model the coupled billiards?

Rare configurations of E can **slow down** mixing of energies and θ dynamics $\Longrightarrow$

- $p(\theta, E)$ may get close to 1 or 0
- Correlation times for $p(\theta, E)$ can blow up

These issues can be studied with the RG

Conclusions

Rather general class of CML with a conservation law can be studied with the RG

A new approach to study hydrodynamic limits of particle systems, interacting random walks etc.

Challenge: real Hamiltonian systems

Linearized RG

Given $p = \{p(t)_{xy}\}$, compute $Rp = \{p'(t')_{x'y'}\}$ from

$$p'(t')_{x'y'} = L^d(p(L^2 t') \dots p(L^2(t'+1)-1))_{Lx'Ly'}$$

Write

$$p_{xy} = T(x-y) + b_{xy}$$

with $Ep = T$ and $Eb = 0$.
Let $Rp = T' + b'$. To linear order in b

$$T'(x'-y') = L^d T^{L^2}(Lx' - Ly')$$

and (let $t' = 0$)

$$b'_{x'y'} = L^d \sum_{t=1}^{L^2} \sum_{xy} T^t(Lx'-x) b_{xy}(t) T^{L^2-t-1}(y - Ly').$$

Contraction

Let $\hat{T}(k) = 1 - ck^2 + o(k^2)$. As $L \to \infty$:

$$\hat{T}'(k) = \hat{T}^{L^2}(k/L) \to e^{-ck^2} = \hat{p}^*(k)$$

For b use $\sum_y p_{xy} = 1$ implying

$$\sum_y b_{xy} = 0$$

to get

$$b'_{x'y'} \sim L^d \sum_{t=1}^{L^2} \sum_{xy} T^t(Lx' - x) b_{xy}(t) \nabla_y T^{L^2-t-1}(y - Ly')$$

Contraction1

For $t = \mathcal{O}(L^2)$,

$$T^t(Lx' - x) \sim L^{-d} e^{-(x' - x/L)^2}$$

$$\nabla_y T^{L^2 - t - 1}(y - Ly') \sim L^{-d-1} e^{-(y' - y/L)^2}$$

so e.g.

$$b'_{00}(0) \sim L^d L^{-d} L^{-d-1} \sum_{t < L^2} \sum_{|x| < L} b_{xx}(t)$$

so since $b_{xx}(t) \sim$ i.i.d.

$$E(Rb)^2 \sim L^{-2d-2} L^{d+2} Eb^2 = L^{-d} Eb^2$$

Noise is **irrelevant** in all dimensions.