Diffusion for coupled maps with a conservation law

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joint work with J. Bricmont

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Heat conduction

Heating

- Contraction1

• **Nonequlibrium stationary state**: finite system, heat bath (noise) on the boundary



- **Approach to equilibrium**: Infinite system, initial state with constant temperature at spatial infinity.
- Models: Coupled maps and Coupled flows
 - Subsystems (maps or flows) indexed by $x \in \mathbb{Z}^d$
 - Couple together locally: system at x interacts with systems at nearby y.

Local energy and flux

Heating

Local energy

Total energy is sum of **local energies** H_x :

$$H = \sum_{x} H_{x}$$

No coupling: each H_x is conserved, $\dot{H}_x = 0$. Turn on coupling: only H is conserved and $\dot{H}_x = -\nabla \cdot J_x$

 J_x flux of energy at site x.

Show: H_x diffuses and J_x is tied to to H_x by Fourier's law.

Return to equilibrium

Contraction

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Return to equilibrium

Infinite system $V = \mathbb{Z}^d$, $\mu_t = \phi_t \mu_0$, ϕ_t Hamiltonian flow Diffusion in hydrodynamic limit:

• Take μ_0 s.t. $E_{\mu_0}H_x = \tau(\epsilon x)$

Let

$$\tau(t, x) := \lim_{\epsilon \to 0} E_{\mu_{t/\epsilon^2}} H_{x/\epsilon}$$
$$j(t, x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_{\mu_{t/\epsilon^2}} J_{x/\epsilon}$$

Show:

 $j = -\kappa(\tau) \nabla \tau$ Fourier law $\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau)$ Diffusion

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Heating Local energy Return to equilibriur

NESS

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NESS

Fix $\Lambda \subset \mathbb{R}^d$ and $t : \partial \Lambda \to \mathbb{R}^+$.

Lattice box $V = \Lambda/\epsilon \cap \mathbb{Z}^d$,

- Hamiltonian dynamics in V
- noise of energy $T_x = t_{x/\epsilon}$ at $x \in \partial V$.

For $x \in \Lambda$ let

$$j_X = rac{1}{\epsilon} E J_{X/\epsilon} \ au_X = E H_{X/\epsilon}$$

E expectation in NESS

Then, as $\epsilon \rightarrow 0$ show

Fourier law: j(x) = -κ(τ(x))∇τ(x)
 Temperature profile:

$$abla \cdot (\kappa(au)
abla au) = 0$$
 $abla |_{\partial \Lambda} = t$

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Heating Local energy Return to equilibrium NESS Coupled dynamics Coupled chaos CMI

CML1

Coupling

Diffusion for maps

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Mixing

Slaving

Diffusion

Scaling limit

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Coupled dynamics

Models: Coupled flows and Coupled maps

1. Weakly anharmonic coupled oscillators:

- At each x ∈ Z^d (an)harmonic oscillator with (an)harmonic interaction with nearest neighbours
- Lots of numerics.
- In a weak anharmonicity scaling limit (kinetic limit) get formally a Boltzman equation (Spohn)
- Diffusion and Fourier proved there (J.B., A.K. math-ph 0703014)
- Hard to prove kinetic limit

2. Weakly coupled chaotic systems

- Coupled billiards or Anosov systems
- Coupled maps with a local conservation law

Heating Local energy Return to equilibriur NESS Coupled dynamics

Coupled chaos CML CML1 Coupling Diffusion for ma RW RW RE Mixing Slaving Diffusion Scaling limit Renormalization Asymptotics Fixed point Semigroup Assumptions

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Coupled chaotic systems

Weakly coupled strongly chaotic flows

 Bunimovich, Liverani, Pellegrinotti, Suhov, Eckmann, Young, Gaspard, Gilbert,....



CMI

Coupled map lattice

Coupled map lattice: phase space $\mathcal{M} = M^{\mathbb{Z}^d}$. **Local dynamics**: $\phi : M \to M$ **Uncoupled** dynamics $\Phi : \mathcal{M} \to \mathcal{M}$ $\Phi(m)_x = \phi(m_x)$ Coupling map $\Psi : \mathcal{M} \to \mathcal{M}$ Ψ local: $\Psi(m)_x$ depends weakly on m_y for |y - x| large. **CML** dynamics: $\Psi \circ \Phi : \mathcal{M} \to \mathcal{M}$

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Coupled map lattice with a conservation law

CMI 1

Dolgopyat, Liverani.....

We take $M = \mathbb{R}_+ \times N$ and

$$\phi(E, \theta) = (E, f(\theta)), \quad E \in \mathbb{R}_+, \ \theta \in N$$

with f hyperbolic, e.g. $N = S^1$, $f(\theta) = 2\theta$, or $N = \mathbb{T}^2$, f torus automorphism

- Energy of each cell is conserved: $E_x \rightarrow E_x$ i.e. one vanishing Lyapunov exponent per unit volume.
- Chaotic dynamics for the rest: $\theta_x \to f(\theta_x)$
- Coupling typically removes the degeneracy
- Look for coupling so that total energy $E = \sum_{x} E_{x}$ is conserved.

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Coupling

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Coupling

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Coupling: nearby cells interact, exchange energy

$$egin{array}{rcl} E_y' &=& \displaystyle{\sum_{|x-y|=1}} p_{xy}(E, heta) E_x \ heta_x' &=& f(heta_x) + g_x(E, heta) \end{array}$$

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p_{xy}, *g_x* depend on *θ_u*, *E_u* for *u* near *x* only
 p_{xy} ≥ 0

•
$$\sum_{y} p_{xy}(E, \theta) = 1$$
 for all E, θ

Total energy $\sum_{x} E_{x}$ conserved.

Diffusion

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Let, at t = 0, $E_x \to T$ as $|x| \to \infty$. Show $E_x(t)$ diffuses to T almost surely in $\theta(0)$

$$E_x(t) - T \sim t^{-d/2} f(x/\sqrt{t})$$

Hydrodynamic scaling limit:

Let E_x(0) = τ(εx)
 Show: lim_{ε→0} E(t/ε², x/ε) = τ(t, x) satisfies

$$\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau)$$

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almost surely in $\theta(0)$.

Random walk

RW

Iteration of coupled maps

$$E_y(t+1) = \sum_{|x-y|=1} p_{xy}(E(t),\theta(t))E_x$$

$$\theta_x(t+1) = f(\theta_x(t)) + g_x(E(t),\theta(t))$$

where $p_{xv} \ge 0$ and

$$\sum_{y} p_{xy}(E,\theta) = 1.$$

• $p_{XV}(E(t), \theta(t)) := p_{XV}(t)$ can be viewed as transition probabilities of a random walk

 \blacktriangleright $E_x(t)$ is (proportional to) the probability of finding the walker at x at time t

Random walk in random environment

- Heating Local energy Return to equilibriur NESS Coupled dynamics Coupled chaos CML CML1 Coupling
- Diffusion for maps RW
- RWRE
- Mixing Slaving Diffusion Scaling limit Renormaliza Asymptotics Fixed point Semigroup Assumptions Result General case
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- Transition probabilities p_{xy}(t) depend on space and time: random walk in a space-time dependent environment
- $p_{xy}(t)$ completely determined by initial conditions of E, θ

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- Prove: walk is diffusive **almost surely** in $\theta|_{t=0}$
- Typical $\theta|_{t=0} \implies \text{random } \rho_{xy}(t)$
- Prove quenched CLT for such walks i.e. a.s. in the p-ensemble

What is the statistics of p like?

Space time mixing environment

Heating Local energy Return to equilibrium NESS Coupled dynamics Coupled chaos CML CML1 Coupling Diffusion for maps RW RWRE Mixing

Slaving Diffusion Scaling limit Renormalization Asymptotics Fixed point Semigroup Assumptions Besult General case Analogy in Contraction Contraction Contraction Contraction

Suppose first p_{xy} and g_x depend **only** on θ i.e.

$$E_{y}(t+1) = \sum_{|x-y|=1} p_{xy}(\theta(t))E_{x}$$

$$\theta_{x}(t+1) = f(\theta_{x}(t)) + g_{x}(\theta(t))$$

Then:

- *f* hyperbolic, , *g* small, smooth ⇒ *f* + *g* hyperbolic
 CML ⇒ θ-dynamics space time mixing
- $\blacktriangleright \implies p_{xy}(t)$ weakly correlated in space and time
- Use Renormalization to prove randomness is irrelevant =>
- Random walk satisfies CLT a.s. in $p \implies$
- *E* diffuses almost surely in $\theta|_{t=0}$ (arxiv 05/08).

Slaving

Slow and fast variables

Suppose p_{xy} depends on *E* too

- θ still space time mixing
- ▶ p(θ(t), E(t)) gets long range correlations through E dependence
- RG \implies *E* dependence **irrelevant** \implies
- CLT still holds (in preparation...)

Suppose also g_x depends on E

Fast variables θ get slaved to the slow ones *E*

Deterministic diffusion is reduced to the study of RW in weakly correlated environment.

Diffusion

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Random walk with space time dependent transition probability $p_{xy}(t)$.

Probability of a walk $\omega = (\omega_0, \dots, \omega_T)$ in time T

$$\mathcal{P}^{\mathcal{T}}(\omega) = \prod_{t=0}^{\mathcal{T}-1} \mathcal{P}_{\omega_t \omega_{t+1}}(t).$$

 E_T expectation in walks with $\omega_0 = 0$. Diffusion constant

$$D_T = T^{-1} E_T \omega(T)^2$$

Diffusion:

$$\lim_{T\to\infty} D_T = D$$

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Scaling limit

Rescale to space Ω of paths $\omega : [0, 1] \to \mathbb{R}^d$

$$\omega(t)=T^{-\frac{1}{2}}\omega_{Tt}$$

 E_T induces expectation \mathcal{E}_T on such paths

$$\mathcal{E}_T F(\omega(\cdot)) = E_T F(T^{-\frac{1}{2}} \omega_{T\cdot})$$

Scaling limit

$$\lim_{T\to\infty}\mathcal{E}_TF:=\mathcal{E}F$$

for $F : \Omega \to \mathbb{R}$ continous on path space. Prove: **allmost surely** in the *p* ensemble \mathcal{E} exists and equals Wiener measure, diffusion constant *D*:

$$D = \lim_{T \to \infty} D_T = \mathcal{E}\omega(1)^2$$

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Probability to walk from x to y during time interval [t, t']: $P_{t,t'}(x, y, p) = (p(t) \dots p(t'-1))_{xy}$

Define renormalized transition probability matrix

$$(R_{l}p)_{xy}(t) = l^{d}P_{l^{2}t,l^{2}(t+1)}(lx,ly,p)$$

for walks on $I^{-1}\mathbb{Z}^d$. Then, if I^2 divides t, t',

Renormalization group

$$P_{t,t'}(x,y,p) = I^{-d} P_{t/l^2,t'/l^2}(I^{-1}x,I^{-1}y,R_lp).$$

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 $R_{l}p$ is the **Renormalization group** flow in a space of random matrices.

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Scaling limit controlled by R_l as $l \to \infty$ • Diffusion constant at time *t*:

$$D(t, p) = t^{-1} \sum_{x} P_{0,t}(0, x, p) x^2$$

reduces to **unit time** one with rates $R_l p$:

$$D(l^2, p) = D(1, R_l p).$$

• Let $F : \Omega \to \mathbb{R}^d$ depend on ω restricted to $\tau^{-1}\mathbb{Z}$ and $l^2 = T/\tau$. Then

$$\mathcal{E}_{T}F(\omega(\cdot)) = E_{\tau}^{R_{l}p}F(\tau^{-\frac{1}{2}}\omega_{\tau})$$

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= fixed time τ problem with rates $R_I p$.

Asymptotics

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Scaling limit controlled by R_l as $l \to \infty$ • Diffusion constant at time *t*:

$$D(t, p) = t^{-1} \sum_{x} P_{0,t}(0, x, p) x^2$$

reduces to **unit time** one with rates $R_l p$:

$$D(l^2,p)=D(1,R_lp).$$

• Let $F : \Omega \to \mathbb{R}^d$ depend on ω restricted to $\tau^{-1}\mathbb{Z}$ and $l^2 = T/\tau$. Then

$$\mathcal{E}_T F(\omega(\cdot)) = E_{\tau}^{R_l p} F(\tau^{-\frac{1}{2}} \omega_{\tau})$$

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= fixed time τ problem with rates $R_l p$.

Fixed point

Fixed point

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If
$$R_I p \to p^*$$
 as $I \to \infty$ then

$$D = D(1, p^*)$$

and scaling limit is given by

$$\mathcal{E}F(\omega(\cdot)) = E_{\tau}^{p^*}F(\tau^{-\frac{1}{2}}\omega_{\tau}).$$

Convergence to Wiener measure:

$$p_{xy}^* = (2\pi D)^{-d/2} e^{-rac{(x-y)^2}{2D}}.$$

Thus, we want to prove $R_l p$ becomes **nonrandom** as $l \rightarrow \infty$ and converges to p^* a.s. in *p*.

Semigroup

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R_l satisfies $R_{ll'} = R_l R_{l'}$.

Study *R_l* **iteratively**:

• Pick
$$L > 1$$
 and let $R := R_L$

• Let
$$p_n = R^n p$$
 i.e. $p_n = R_{L^n} p$

▶ Let *E* be expectation in *p* ensemble. Write

$$p_n = Ep_n + b_n$$

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Show

• $b_n \rightarrow 0$ almost surely • $Ep_n \rightarrow p^*$.

Assumptions

Assume

- Distribution of p translationally and rotationally invariant
- $Ep_{xy} = T(x y)$ exponentially decaying
- Cumulants of p cluster exponentially

$$E(\rho_{x_1y_1}(t_1); \rho_{x_2y_2}(t_2); \ldots; \rho_{x_Ny_N}(t_N)) \leq \epsilon^N e^{-\lambda \tau},$$

 τ length of shortest tree on the space time support Assumptions are satisfied by

- $p(\theta)$ analytic, local with θ analytic CML
- p(s) local in spins of a high temeperature Ising model and the like

Assumptions

Result

Result

Theorem: In all dimensions $d \ge 1$

$$E(R^{n}p_{x_{1}y_{1}}(t_{1}); R^{n}p_{x_{2}y_{2}}(t_{2}); \ldots; R^{n}p_{x_{N}y_{N}}(t_{N})) \leq \epsilon_{n}^{N}e^{-\lambda\tau},$$

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with $\epsilon_n \to 0$ as $n \to \infty$ (exponentially).

Randomness irrelevant in all dimensions Implications: D exists, scaling limit Wiener

General case

General case

Include *E* dependence of the walk

$$E_y(t+1) = \sum_{|x-y|=1} p(\theta(t), E(t))_{xy} E_x(t)$$

Environment depends on the trajectory E(t). p(s), p(t) diffusively correlated.

RG for **conditional** transition probabilities.

• Let
$$E_n(t, x) = L^{nd}E(L^{2n}t, L^nx)$$

- ▶ p_n(t) = p_n(t, E_n(t)) is conditioned on E_n(t) i.e. collects rescaled walks on time interval [L²ⁿt, L²ⁿ(t + 1)] conditioned on E(L²ⁿt).
- { $p_n(t, E)$ }, *E* fixed **exponentially** weakly correlated
- E dependence an irrelevant perturbation in the RG.

Analogy in continuum

Analogy in continuum

Contraction

Contraction1

Our CML is a discrete version of the SPDE

$$\dot{m{E}}=\partial_{\mu}(m{a}_{\mu
u}(m{E})\partial_{
u}m{E}+m{b}_{\mu}(m{E})m{E})$$

where $a_{\mu\nu}$ and b_{μ} are random and nonlinear The RG produces a non random PDE in the scaling limit

$$\dot{E} = \partial_{\mu}(\kappa_{\mu
u}(E)\partial_{
u}E)$$

It is a combination of RG for PDE's (J.B. & A.K., 1992) and RG for RWRE (J.B. & A.K., 1991).

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However, the randomness is deterministic

Hamiltonian systems

Hamiltonian systems

Conclusions Linearized RG Contraction What kind of CML should model the coupled billiards? **Rare configurations** of *E* can **slow down** mixing of energies and θ dynamics \Longrightarrow

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- $p(\theta, E)$ may get close to 1 or 0
- Correlation times for $p(\theta, E)$ can blow up

These issues can be studied with the RG

Conclusions

Conclusions

Linearized RG Contraction Rather general class of CML with a conservation law can be studied with the RG

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A new approach to study hydrodynamic limits of particle systems, interacting random walks etc.

Challenge: real Hamiltonian systems

Linearized RG

Given $p = \{p(t)_{xy}\}$, compute $Rp = \{p'(t')_{x'y'}\}$ from $p'(t')_{x'y'} = L^d(p(L^2t')\dots p(L^2(t'+1)-1))_{x'y'}$

Write

$$p_{xy}=T(x-y)+b_{xy}$$

with Ep = T and Eb = 0.

Let Rp = T' + b'. To linear order in b

$$T'(x'-y')=L^d T^{L^2}(Lx'-Ly')$$

and (let t' = 0)

Linearized BG

$$b'_{x'y'} = L^d \sum_{t=1}^{L^2} \sum_{xy} T^t (Lx'-x) b_{xy}(t) T^{L^2-t-1}(y-Ly').$$

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Contraction

Local energy NESS Diffusion for maps Contraction

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Let
$$\hat{T}(k) = 1 - ck^2 + o(k^2)$$
. As $L \to \infty$:
 $\hat{T}'(k) = \hat{T}^{L^2}(k/L) \to e^{-ck^2} = \hat{p}^*(k)$
For *b* use $\sum_y p_{xy} = 1$ implying
 $\sum_y b_{xy} = 0$

to get

$$b'_{x'y'} \sim L^d \sum_{t=1}^{L^2} \sum_{xy} T^t (Lx'-x) b_{xy}(t) \nabla_y T^{L^2-t-1}(y-Ly')$$

Contraction1

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Contraction1

For
$$t = \mathcal{O}(L^2)$$
,
 $T^t(Lx' - x) \sim L^{-d} e^{-(x' - x/L)^2}$
 $\nabla_y T^{L^2 - t - 1}(y - Ly') \sim L^{-d - 1} e^{-(y' - y/L)^2}$
so e.g.
 $b'_{00}(0) \sim L^d L^{-d} L^{-d - 1} \sum_{t < L^2} \sum_{|x| < L} b_{xx}(t)$

so since $b_{xx}(t) \sim \text{i.i.d.}$

$$E(Rb)^2 \sim L^{-2d-2}L^{d+2}Eb^2 = L^{-d}Eb^2$$

Noise is irrelevant in all dimensions.