

Transport Properties of a Chain of Quantum Dots with Self-Consistent Reservoirs

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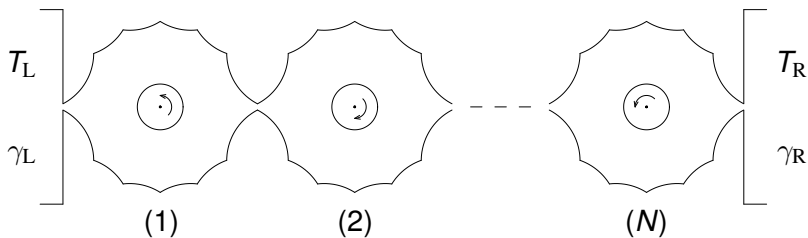
Vienna, 6th June 2008

Collaborators: M. Büttiker, J.-P. Eckmann and C. Mejía Monasterio

Outline

- 1 Introduction
 - Motivations
 - The Model
 - The Strategy
- 2 General Properties
 - The Electric and Heat Currents
 - Onsager Relations and Entropy Production
 - Equilibrium and Non-Equilibrium States
- 3 Ohm and Fourier Laws
 - The Self-Consistency Condition
 - Simulations (RMT)
- 4 Conclusion

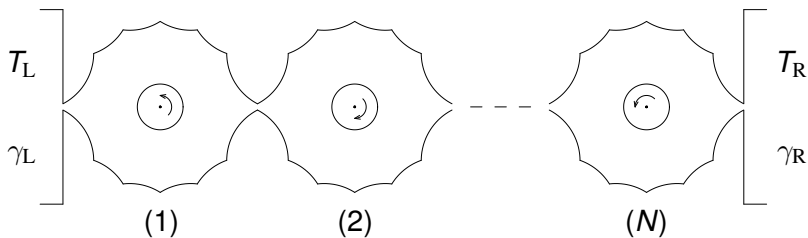
The classical EY-model



Ref. J.-P. Eckmann and L.-S. Young *Commun. Math. Phys.* **262** 237 (2006)

Ref. H. Larralde, F. Leyvraz and C. Mejía-Monasterio *J. Stat. Phys.* **113** 197 (2003)

The classical EY-model



Assumption: The system admits a stationary state

Fourier's law

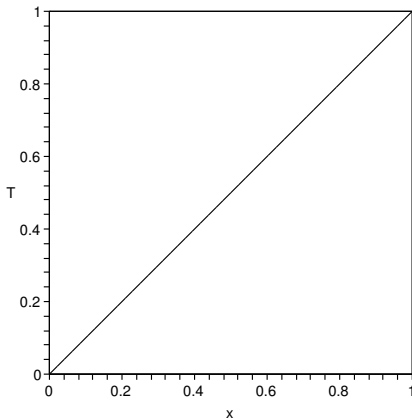
$$J = -\gamma \frac{T_R - T_L}{N} \quad (\gamma_L = \gamma_R = \gamma)$$

Ref. J.-P. Eckmann and L.-S. Young *Commun. Math. Phys.* **262** 237 (2006)

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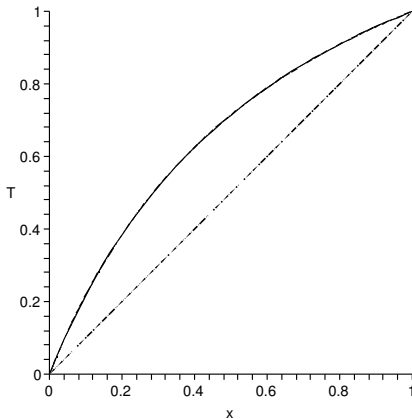
The classical EY-model

Temperature profile of the discs



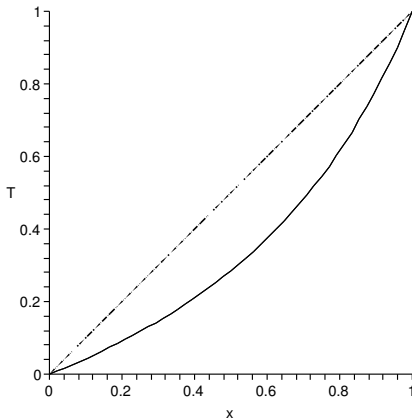
The classical EY-model

Temperature profile of the discs

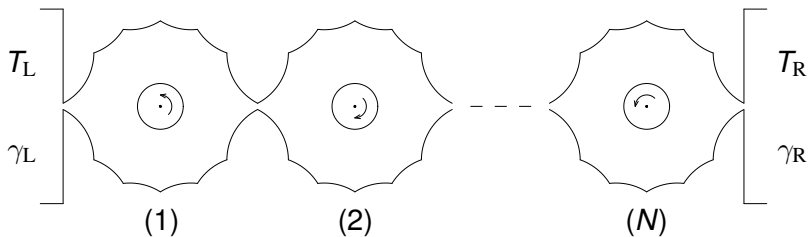


The classical EY-model

Temperature profile of the discs



The classical EY-model



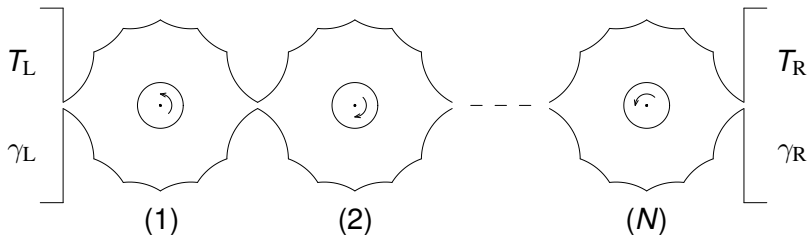
The problem

Establish a **quantum version** of the EY-model

Questions

- 1 Does Fourier's law still hold ?
- 2 Interference effects on the profile and current ?

The classical EY-model



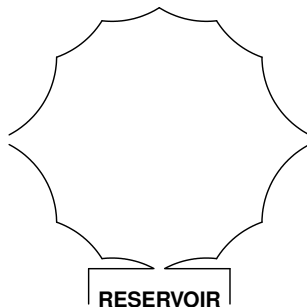
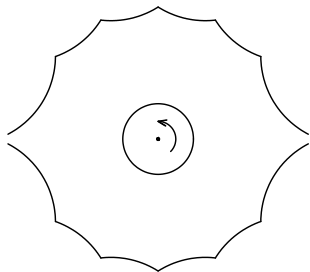
The problem

Establish a **quantum version** of the EY-model

Difficulty

What are quantum discs ?

An Effective Disc



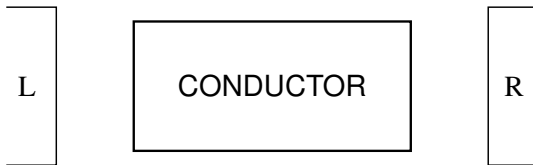
Idea

Effective disc = Particle reservoir satisfying the **self-consistency condition**^a

$$I = 0 \quad \text{and} \quad J = 0$$

^aRef. M. Visscher and M. Rich *PRA* **12** 675 (1975)

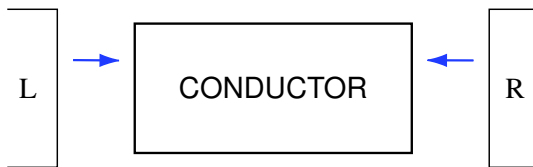
The Scattering Approach



Idea

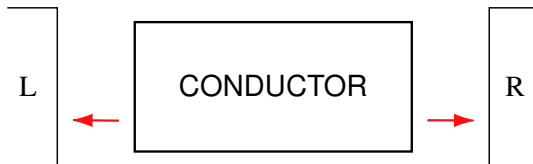
View the CONDUCTOR as a TARGET for the particles

The Scattering Approach



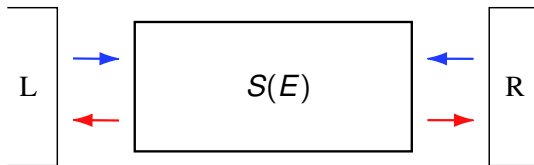
$$\psi^{\text{in}}(E)$$

The Scattering Approach



$$\psi^{\text{out}}(E)$$

The Scattering Approach



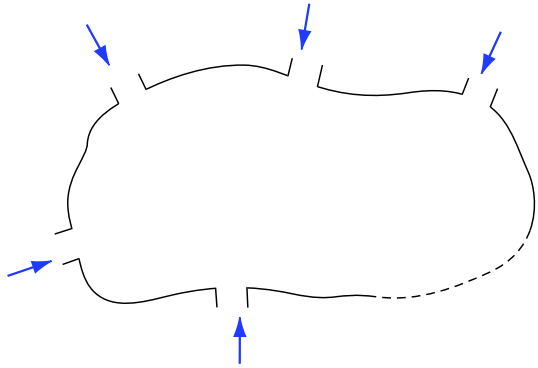
Scattering Matrix

$$\psi^{\text{out}}(E) = S(E) \psi^{\text{in}}(E)$$

Particle current conservation needs $S(E)$ to be **unitary**

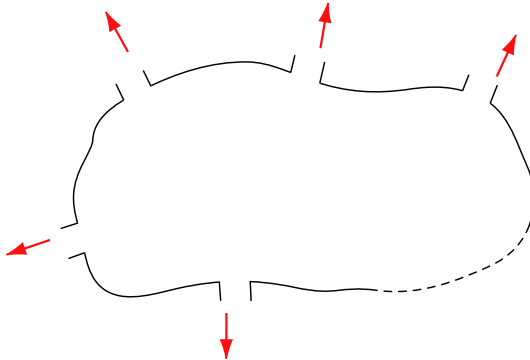
The Scattering Approach

Multi-lead system



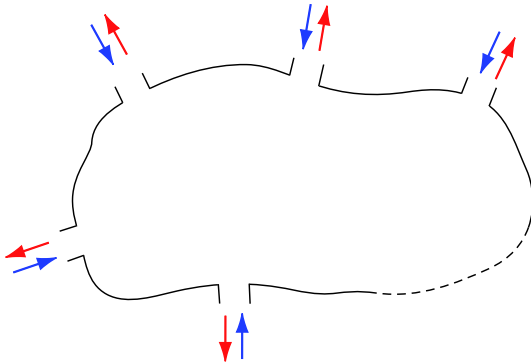
The Scattering Approach

Multi-lead system



The Scattering Approach

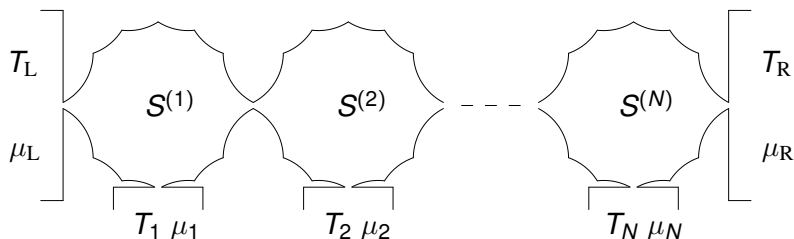
Multi-lead system



Scattering Matrix

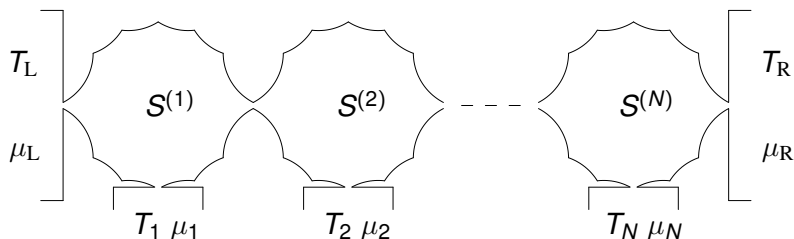
$$\psi^{\text{out}}(E) = S(E) \psi^{\text{in}}(E)$$

A Chain of Quantum Dots



- 1 Transport properties: Scattering matrices $S^{(1)}, \dots, S^{(N)}$

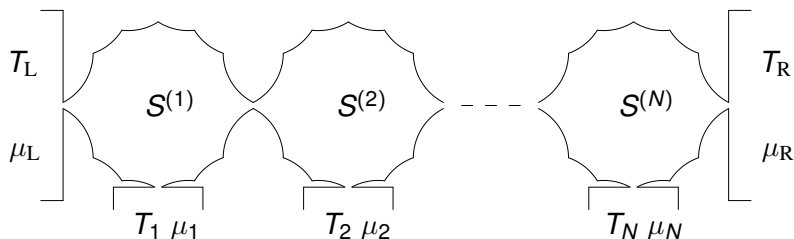
A Chain of Quantum Dots



- 1 Transport properties: Scattering matrices $S^{(1)}, \dots, S^{(N)}$
- 2 Effective discs = Particle reservoirs satisfying the **self-consistency condition**

$$I_i = 0 \quad \text{and} \quad J_i = 0 \quad \text{for} \quad i = 1, \dots, N$$

A Chain of Quantum Dots

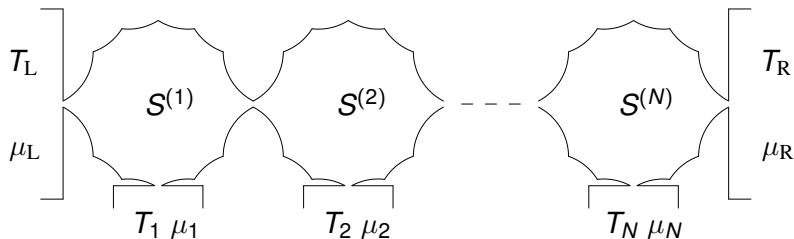


- 1 Transport properties: Scattering matrices $S^{(1)}, \dots, S^{(N)}$
- 2 Effective discs = Particle reservoirs satisfying the **self-consistency condition**

$$I_i = 0 \quad \text{and} \quad J_i = 0 \quad \text{for} \quad i = 1, \dots, N$$

- 3 Reservoirs : MB, FD or BE

A Chain of Quantum Dots

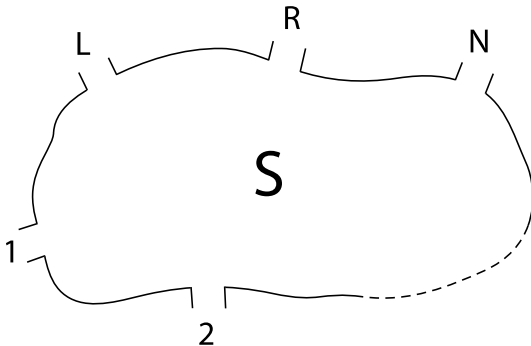


Terminology

- All expressions without a superscript (MB, FD or BE) hold in the three cases
- Universality = Independent of f (MB, FD or BE)

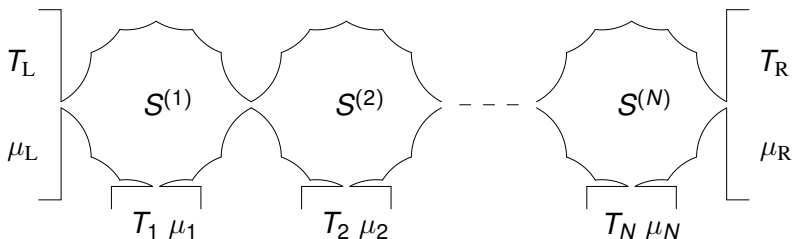
PART 1 : Any Geometry

- Multi-lead system with $N + 2$ leads and **any** scattering matrix S



- Introduce the **framework** and present the **main properties** in the **linear response regime**

PART 2 : Linear Geometry



- Given (T_L, μ_L) and (T_R, μ_R) we find (T_i, μ_i) , for $i = 1 \dots, N$, satisfying the **self-consistency condition**

$$I_i = 0 \quad \text{and} \quad J_i = 0 \quad \text{for} \quad i = 1, \dots, N$$

- Determine the currents: $I_L = -I_R$ and $J_L = -J_R$
- Linear geometry : Build S from $S^{(1)}, \dots, S^{(N)}$
 $\implies I_L$ and J_L go from left to right

The Electric Current

Notations ($i, j \in \{L, 1, \dots, N, R\}$)

- The scattering matrix: S
- The transmission probability: $t_{ij} = |S_{ij}|^2$
- The distribution function

$$f_i(E) = \underbrace{\exp\left(-\frac{E - \mu_i}{k_B T_i}\right)}_{MB} \text{ or } \underbrace{\left[\exp\left(\frac{E - \mu_i}{k_B T_i}\right) \pm 1\right]^{-1}}_{\substack{+ : FD \\ - : BE}}$$

- Boltzmann and Planck constants: k_B, h Charge: e

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- Boltzmann and Planck constants: k_B, h Charge: e

Assumption: No interaction among the particles

$$I_i = \frac{e}{h} \sum_{j \neq i} \int_0^\infty [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] dE$$

Ref. Electronic Transport in Mesoscopic Systems by S. Datta (1995)

The Heat Current

$$(\text{Particle Current})_i = \frac{1}{h} \sum_{j \neq i} \int_0^{\infty} [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] dE$$

$$(\text{Energy Current})_i = \frac{1}{h} \sum_{j \neq i} \int_0^{\infty} [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] E dE$$

The Heat Current

$$(\text{Particle Current})_i = \frac{1}{h} \sum_{j \neq i} \int_0^\infty [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] dE$$

$$(\text{Energy Current})_i = \frac{1}{h} \sum_{j \neq i} \int_0^\infty [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] E dE$$

First Law of Thermodynamics: $\delta Q = dE - \mu dN$

$$J_i = \frac{1}{h} \sum_{j \neq i} \int_0^\infty [f_i(E)t_{ji}(E) - f_j(E)t_{ij}(E)] (E - \mu_i) dE$$

Ref. P. N. Butcher *J. Phys.: Condens. Matter* **2** (1990)

Summary

$$\Gamma_{ij} = \delta_{ij} - t_{ij} \quad t_{ij} = |S_{ij}|^2$$

Electric Current: $I_i = \frac{e}{h} \sum_j \int_0^\infty f_j(E) \Gamma_{ij}(E) dE$

Heat Current: $J_i = \frac{1}{h} \sum_j \int_0^\infty f_j(E) \Gamma_{ij}(E) (E - \mu_i) dE$

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Conservation laws

Charge: $\sum_i I_i = 0$

Energy: $\sum_i \left(J_i + \frac{\mu_i}{e} I_i \right) = 0$

Summary

$$\Gamma_{ij} = \delta_{ij} - t_{ij} \quad t_{ij} = |S_{ij}|^2$$

Electric Current: $I_i = \frac{e}{h} \sum_j \int_0^\infty f_j(E) \Gamma_{ij}(E) dE$

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Conservation laws

Charge: $\sum_i I_i = 0$

Heat in Linear Regime: $\sum_i J_i = 0$

Linear Response

Notations: $T_j = T + \delta T_j$ and $\mu_j = \mu + \delta\mu_j$

$$f(E; T_j, \mu_j) = f(E; T, \mu) + \frac{\partial f}{\partial T}(E; T, \mu) \delta T_j + \frac{\partial f}{\partial \mu}(E; T, \mu) \delta \mu_j$$

Linear Response

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For f MB, FD or BE

$$f(E; T_j, \mu_j) = f\left(\frac{E - \mu_j}{k_B T_j}\right)$$

Linear Response

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$$f(E; T_j, \mu_j) = f(E; T, \mu) + \underbrace{\frac{\partial f}{\partial T}(E; T, \mu)}_{-\frac{\partial f}{\partial E}(E; T, \mu) \left(\frac{E-\mu}{T}\right)} \delta T_j + \underbrace{\frac{\partial f}{\partial \mu}(E; T, \mu)}_{-\frac{\partial f}{\partial E}(E; T, \mu)} \delta \mu_j$$

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For f MB, FD or BE

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For f MB, FD or BE

$$f(E; T_j, \mu_j) = f(E; T, \mu) - \frac{\partial f}{\partial E}(E; T, \mu) \left[\left(\frac{E - \mu}{T}\right) \delta T_j + \delta \mu_j \right]$$

Linear Response

$$I_i = \frac{e}{h} \sum_j \int_0^\infty f_j(E) \Gamma_{ij}(E) dE$$

$$J_i = \frac{1}{h} \sum_j \int_0^\infty f_j(E) \Gamma_{ij}(E) (E - \mu_i) dE$$

$$f(E; T_j, \mu_j) = f(E; T, \mu) - \frac{\partial f}{\partial E}(E; T, \mu) \left[\left(\frac{E - \mu}{T} \right) \delta T_j + \delta \mu_j \right]$$

Onsager Relations

$$I_i = \sum_j L_{ij}^{(0)} \frac{\delta\mu_j}{e} + L_{ij}^{(1)} \frac{\delta T_j}{T}$$

$$J_i = \sum_j L_{ij}^{(1)} \frac{\delta\mu_j}{e} + L_{ij}^{(2)} \frac{\delta T_j}{T}$$

$$L_{ij}^{(0)} = -\frac{e^2}{h} \int_0^\infty \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

$$L_{ij}^{(1)} = -\frac{e}{h} k_B T \int_0^\infty \left(\frac{E - \mu}{k_B T} \right) \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

$$L_{ij}^{(2)} = -\frac{(k_B T)^2}{h} \int_0^\infty \left(\frac{E - \mu}{k_B T} \right)^2 \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

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$$L_{ij}^{(0)} = -\frac{e^2}{h} \int_0^\infty \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE \quad \Gamma_{ij} = \delta_{ij} - |S_{ij}|^2$$

$$L_{ij}^{(1)} = -\frac{e}{h} k_B T \int_0^\infty \left(\frac{E - \mu}{k_B T} \right) \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

$$L_{ij}^{(2)} = -\frac{(k_B T)^2}{h} \int_0^\infty \left(\frac{E - \mu}{k_B T} \right)^2 \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

The Onsager relations hold: $S_{ij} = S_{ji} \implies L_{ij} = L_{ji}$

The Main Assumption

Assumption

S does not depend on the energy

Interests

- Good approximation in some limit cases (e.g. low T in FD)
- Consequences of $S(E) = \text{Constant}$

Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = -\frac{e^2}{h} \int_0^\infty \frac{\partial f}{\partial E}(E; T, \mu) \Gamma_{ij}(E) dE$$

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Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = -\frac{e^2}{h} \int_0^\infty \frac{\partial f}{\partial E}(E; T, \mu) dE \cdot \Gamma_{ij}$$

$$L_{ij}^{(1)} = -\frac{e}{h} k_B T \int_0^\infty \left(\frac{E - \mu}{k_B T} \right) \frac{\partial f}{\partial E}(E; T, \mu) dE \cdot \Gamma_{ij}$$

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Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = \frac{e^2}{h} \underbrace{\left[- \int_0^\infty \frac{\partial f}{\partial E}(E; T, \mu) dE \right]}_{C(0)} \cdot \Gamma_{ij}$$

$$L_{ij}^{(1)} = \frac{e}{h} k_B T \underbrace{\left[- \int_0^\infty \left(\frac{E - \mu}{k_B T} \right) \frac{\partial f}{\partial E}(E; T, \mu) dE \right]}_{C(1)} \cdot \Gamma_{ij}$$

$$L_{ij}^{(2)} = \frac{(k_B T)^2}{h} \underbrace{\left[- \int_0^\infty \left(\frac{E - \mu}{k_B T} \right)^2 \frac{\partial f}{\partial E}(E; T, \mu) dE \right]}_{C(2)} \cdot \Gamma_{ij}$$

Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = \frac{e^2}{h} C(0) \Gamma_{ij}$$

$$L_{ij}^{(1)} = \frac{e}{h} k_B T C(1) \Gamma_{ij}$$

$$L_{ij}^{(2)} = \frac{(k_B T)^2}{h} C(2) \Gamma_{ij}$$

Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = \frac{e^2}{h} C(0) \Gamma_{ij}$$

$$L_{ij}^{(1)} = \frac{e}{h} k_B T C(1) \Gamma_{ij} = \frac{k_B T}{e} \frac{C(1)}{C(0)} L_{ij}^{(0)}$$

$$L_{ij}^{(2)} = \frac{(k_B T)^2}{h} C(2) \Gamma_{ij} = \frac{k_B^2 T^2}{e^2} \frac{C(2)}{C(0)} L_{ij}^{(0)}$$

Consequence of $S(E) = \text{Constant}$

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The Coefficients $C(n)$

$$C(n) = - \int_0^\infty \left(\frac{E - \mu}{k_B T} \right)^n \frac{\partial f}{\partial E}(E; T, \mu) dE$$

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$$C(n) = - \int_0^{\infty} \left(\frac{E - \mu}{k_B T} \right)^n \frac{\partial f}{\partial E}(E; T, \mu) dE$$

$$C^{\text{MB}}(n) = \int_{x_0}^{\infty} x^n e^{-x} dx$$

$$C^{\pm}(n) = \int_{x_0}^{\infty} \frac{x^n e^x}{(e^x \pm 1)^2} dx$$

where

$$x_0 = -\frac{\mu}{k_B T} \begin{cases} \in (-\infty, \infty) \text{ in MB} \\ \in (-\infty, \infty) \text{ in FD} \\ \in (0, \infty) \text{ in BE} \end{cases}$$

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We need $C(n) > 0$ for $n = 0, 1, 2$

Summary

$$I_i = \sum_j L_{ij}^{(0)} \frac{\delta\mu_j}{e} + L_{ij}^{(1)} \frac{\delta T_j}{T}$$

$$J_i = \sum_j L_{ij}^{(1)} \frac{\delta\mu_j}{e} + L_{ij}^{(2)} \frac{\delta T_j}{T}$$

Consequence of $S(E) = \text{Constant}$

$$L_{ij}^{(0)} = \frac{e^2}{h} C(0) \Gamma_{ij} \quad L_{ij}^{(1)} = \tilde{L}_0 L_{ij}^{(0)} \quad \text{and} \quad L_{ij}^{(2)} = L_0 L_{ij}^{(0)}$$

$$L_0 = \frac{k_B^2 T^2}{e^2} \frac{C(2)}{C(0)} > 0 \quad \text{and} \quad \tilde{L}_0 = \frac{k_B T}{e} \frac{C(1)}{C(0)} > 0$$

The Transport Matrix

$$L_{ij}^{(k)} \sim \Gamma_{ij} = \delta_{ij} - t_{ij} \quad (t_{ij} \in (0, 1))$$

Properties

1

$$L_{ij}^{(k)} \begin{cases} > 0 & \text{if } i = j \\ < 0 & \text{if } i \neq j \end{cases}$$

2

$$\sum_i L_{ij}^{(k)} = 0, \quad \forall j \quad \text{and} \quad \sum_j L_{ij}^{(k)} = 0, \quad \forall i$$

The Transport Matrix

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$$L_{ij}^{(k)} \begin{cases} > 0 & \text{if } i = j \\ < 0 & \text{if } i \neq j \end{cases}$$

2

$$\sum_i L_{ij}^{(k)} = 0, \quad \forall j \quad \text{and} \quad \sum_j L_{ij}^{(k)} = 0, \quad \forall i$$

For f MB, FD or BE ($L_{ij}^{(1)} = \tilde{L}_0 L_{ij}^{(0)}$ and $L_{ij}^{(2)} = L_0 L_{ij}^{(0)}$)

$$\mathcal{R} \equiv \frac{\tilde{L}_0}{\sqrt{L_0}} = \frac{C(1)}{\sqrt{C(0) \cdot C(2)}} \in (0, 1), \quad \forall x_0 = -\mu/(k_B T)$$

Entropy Production

Theorem

$$L = \begin{pmatrix} L^{(0)} & L^{(1)} \\ L^{(1)} & L^{(2)} \end{pmatrix}, \quad L^{(k)} = \begin{pmatrix} L_{LL}^{(k)} & L_{L1}^{(k)} & \cdots & L_{LN}^{(k)} & L_{LR}^{(k)} \\ L_{1L}^{(k)} & L_{11}^{(k)} & \cdots & L_{1N}^{(k)} & L_{1R}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{NL}^{(k)} & L_{N1}^{(k)} & \cdots & L_{NN}^{(k)} & L_{NR}^{(k)} \\ L_{RL}^{(k)} & L_{R1}^{(k)} & \cdots & L_{RN}^{(k)} & L_{RR}^{(k)} \end{pmatrix}$$

The matrix L is real positive semi-definite:

$$\sigma_s = \sum_i \sum_j L_{ij} V_i V_j \geq 0$$

$$V_i = \begin{cases} \delta\mu_i/e & \text{if } i = 1, \dots, N+2 \\ \delta T_i/T & \text{if } i = N+3, \dots, 2N+4 \end{cases}$$

Main Trick of the Proof $\sigma_s \geq 0$

Notations: $X_i = \delta\mu_i/e$ and $Z_i = \sqrt{L_0} \delta T_i/T$

$$\sigma_s = \sum_{i < j} \underbrace{(-L_{ij}^{(0)})}_{>0} I_{ij}$$

$$I_{ij} = (X_i - X_j)^2 + (Z_i - Z_j)^2 - 2\mathcal{R}C_{ij}, \quad C_{ij} = X_i Z_j + X_j Z_i - X_i Z_i - X_j Z_j$$

TRICK : $0 < \mathcal{R} < 1$

$$C_{ij} \leq 0 : I_{ij} \geq (X_i - X_j)^2 + (Z_i - Z_j)^2 \geq 0$$

$$C_{ij} > 0 : I_{ij} > (X_i - X_j)^2 + (Z_i - Z_j)^2 - 2C_{ij} = (X_i - X_j + Z_i - Z_j)^2 \geq 0$$

Ref. M. Büttiker *IBM J. Res. Dev.* **3** 317-334 (1988)

Equilibrium and Non-Equilibrium States

Definition (Equilibrium)

$$\mu_L = \mu_1 = \dots = \mu_N = \mu_R \quad \text{and} \quad T_L = T_1 = \dots = T_N = T_R$$

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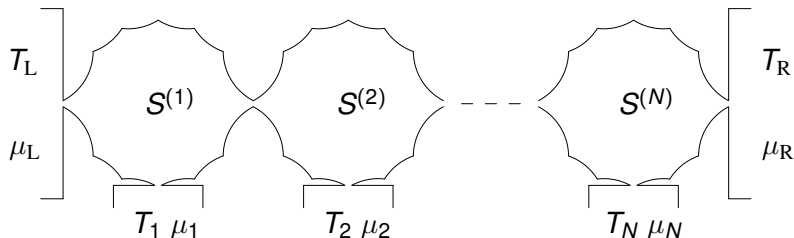
Theorem

$$\left\{ \begin{array}{l} \text{System is at} \\ \text{equilibrium} \end{array} \right\} \iff \sigma_s = 0$$

$$\left\{ \begin{array}{l} \text{System is} \\ \text{out of equilibrium} \end{array} \right\} \iff \sigma_s > 0$$

Main trick in the proofs: $0 < \mathcal{R} < 1$

The Self-Consistency Condition



The problem

Given (T_L, μ_L) and (T_R, μ_R) , find (T_i, μ_i) , for $i = 1 \dots, N$, such that

$$I_i = 0 \quad \text{and} \quad J_i = 0 \quad \text{for} \quad i = 1, \dots, N$$

The Self-Consistency Condition

For $i = 1, \dots, N$:

$$I_i = \sum_j L_{ij}^{(0)} \frac{\delta\mu_j}{e} + L_{ij}^{(1)} \frac{\delta T_j}{T} = 0$$

$$J_i = \sum_j L_{ij}^{(1)} \frac{\delta\mu_j}{e} + L_{ij}^{(2)} \frac{\delta T_j}{T} = 0$$

$$\sum_{j=1}^N \left(L_{ij}^{(0)} \frac{\delta\mu_j}{e} + L_{ij}^{(1)} \frac{\delta T_j}{T} \right) = - \sum_{j=L,R} \left(L_{ij}^{(0)} \frac{\delta\mu_j}{e} + L_{ij}^{(1)} \frac{\delta T_j}{T} \right)$$

$$\sum_{j=1}^N \left(L_{ij}^{(1)} \frac{\delta\mu_j}{e} + L_{ij}^{(2)} \frac{\delta T_j}{T} \right) = - \sum_{j=L,R} \left(L_{ij}^{(1)} \frac{\delta\mu_j}{e} + L_{ij}^{(2)} \frac{\delta T_j}{T} \right)$$

The Self-Consistency Condition

Notations: $X_i = \delta\mu_i/e$, $Y_i = \delta T_i/T$, $M_{ij} = L_{ij}^{(0)}$, $(D_\ell)_i = L_{i\ell}^{(0)}$

$$M(X + \tilde{L}_0 Y) = - \sum_{\ell=L,R} (X_\ell + \tilde{L}_0 Y_\ell) D_\ell$$

$$M(\tilde{L}_0 X + L_0 Y) = - \sum_{\ell=L,R} (\tilde{L}_0 X_\ell + L_0 Y_\ell) D_\ell$$

TRICK: $\mathcal{R} \neq 1 \implies L_0 \neq (\tilde{L}_0)^2$

$$MX = - \sum_{\ell=L,R} D_\ell X_\ell \quad \text{and} \quad MY = - \sum_{\ell=L,R} D_\ell Y_\ell$$

The profiles

Theorem ($i = 1, \dots, N$, $\Gamma_{ij} = \delta_{ij} - t_{ij}$)

$$\mu_i = \mu_L + A_i(\mu_R - \mu_L)$$

$$T_i = T_L + A_i(T_R - T_L) \quad \text{with} \quad A_i = \sum_{j=1}^N (\Gamma^{-1})_{ij} t_{jR}$$

Consequences of $S(E) = \text{Constant}$

- The profiles of μ and T are decoupled \neq EY-model
- The profiles are **universal**: they do not depend on f
- The profiles are given by A_1, \dots, A_N

The profiles

Lemma ($i = 1, \dots, N, \Gamma_{ij} = \delta_{ij} - t_{ij}$)

We have

$$A_i = \frac{\sum_{j=1}^N (-1)^{i+j} \det(\Gamma(j, i)) t_{jR}}{\sum_{j=1}^N (-1)^{i+j} \det(\Gamma(j, i)) [t_{jL} + t_{jR}]}$$

where $\Gamma(j, i)$ is the (j, i) minor of Γ

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where $\Gamma(j, i)$ is the (j, i) minor of Γ

Example ($N = 1$)

$$A_1 = \frac{t_{1R}}{t_{1L} + t_{1R}} \quad \implies \quad \mu_1 = \frac{t_{1L}\mu_L + t_{1R}\mu_R}{t_{1L} + t_{1R}}$$

Ref. M. Büttiker *IBM J. Res. Dev.* **3** 317-334 (1988)

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where $\Gamma(j, i)$ is the (j, i) minor of Γ

Consequence: $\mu_L < \mu_i < \mu_R$ and $T_L < T_i < T_R$

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$$A_1 = \frac{t_{1R}}{t_{1L} + t_{1R}} \quad \Rightarrow \quad \mu_1 = \frac{t_{1L}\mu_L + t_{1R}\mu_R}{t_{1L} + t_{1R}}$$

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The currents

Theorem ($i = 1, \dots, N, \Gamma_{ij} = \delta_{ij} - t_{ij}$)

$$I_L = \sigma_0 \left(\frac{\mu_R - \mu_L}{e} \right) + \sigma_1 \left(\frac{T_R - T_L}{T} \right)$$

$$J_L = \sigma_1 \left(\frac{\mu_R - \mu_L}{e} \right) + \sigma_2 \left(\frac{T_R - T_L}{T} \right)$$

$$\sigma_0 = L_{LR}^{(0)} + \sum_{j=1}^N A_j L_{Lj}^{(0)}, \quad \sigma_1 = \tilde{L}_0 \sigma_0 \quad \text{and} \quad \sigma_2 = L_0 \sigma_0$$

Remarks

$$L_{ij}^{(1)} = \tilde{L}_0 L_{ij}^{(0)} \quad \text{and} \quad L_{ij}^{(2)} = L_0 L_{ij}^{(0)} \quad \sigma_k < 0$$

The currents

Ohm's law ($T_L = T_R, \mu = eV$):

$$I_L = -\kappa_e \frac{V_R - V_L}{N}, \quad \kappa_e = -N\sigma_0 > 0$$

Fourier's law ($I_L = 0$):

$$J_L = -\kappa_h \frac{T_R - T_L}{N}, \quad \kappa_h = N \frac{\sigma_1^2 - \sigma_0 \sigma_2}{\sigma_0 T} = -\frac{(L_0 - \tilde{L}_0^2)}{T} N \sigma_0$$

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Universal conductivity ($\Gamma_{ij} = \delta_{ij} - t_{ij}$)

$$\sigma(N) = -\frac{h}{e^2 C(0)} N \sigma_0 = N \left[t_{LR} + \sum_{i,j=1}^N t_{Lj} (\Gamma^{-1})_{ji} t_{iR} \right]$$

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Random Matrix Theory

Question

For which $\{S_N\}_{N=1}^{\infty}$ does the limit $\lim_{N \rightarrow \infty} \sigma(N)$ exist ?

Assumption

The 3×3 complex matrices $S^{(1)}, \dots, S^{(N)}$ are independent and identically distributed over $U(3)$ with some measure:

- 1 **COE** = Circular Orthogonal Ensemble
Time-reversible $S_{ij}^{(k)} = S_{ji}^{(k)}$
- 2 **CUE** = Circular Unitary Ensemble
Not time-reversal $S_{ij}^{(k)} \neq S_{ji}^{(k)}$ (e.g. magnetic field)

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Quantum versus Classical

Universality

A_1, \dots, A_N and $\sigma(N)$ do not depend on f

Interference effects ?

Quantum

$\varphi_1, \varphi_2 \in \mathbb{C} \implies P = |\varphi_1 + \varphi_2|^2 = |\varphi_1|^2 + |\varphi_2|^2 + \text{Interferences}$

Classical

$\varphi_1, \varphi_2 \in \mathbb{C} \implies P = |\varphi_1|^2 + |\varphi_2|^2$

Quantum versus Classical

Universality

A_1, \dots, A_N and $\sigma(N)$ do not depend on f

Interference effects ?

Quantum

$$S^{(1)}, \dots, S^{(N)} \implies S \implies t_{ij} = |S_{ij}|^2$$

Classical

$$S^{(1)}, \dots, S^{(N)} \implies P^{(1)}, \dots, P^{(N)} \text{ where } P_{ij}^{(k)} = |S_{ij}^{(k)}|^2 \\ \implies P \implies t_{ij} = P_{ij}$$

The Statistical Averages COE and CUE

- 1 The Average Transmission Probabilities $\langle t_{ij} \rangle$
- 2 The Average Universal Conductivity $\langle \sigma(N) \rangle$
- 3 The Average Universal Profiles $\langle A_1 \rangle, \dots, \langle A_N \rangle$

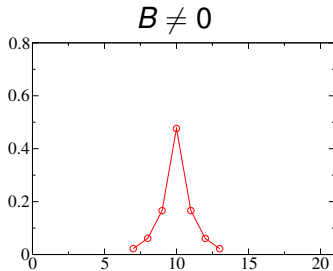
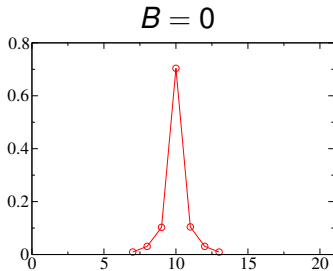
The Average Transmission Probabilities $\langle t_{ij} \rangle$

When $N \gg 1$

- 1 $t_{ij}^{cl} \neq t_{ij}^{qu}$ but $\langle t_{ij}^{cl} \rangle = \langle t_{ij}^{qu} \rangle$
- 2 $\langle t_{ij} \rangle$ do not depend on N
- 3 Symmetric: $\langle t_{ij} \rangle = \langle t_{ji} \rangle$
- 4 $\langle t_{ij} \rangle$ depend on $|i - j|$

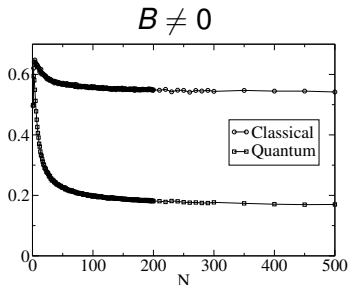
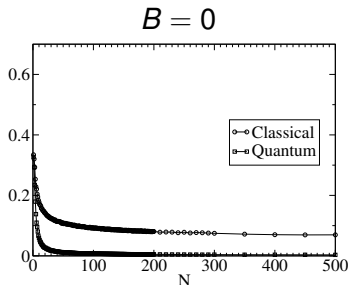
The Average Transmission Probabilities $\langle t_{ij} \rangle$

- 1 Short range: $\langle t_{ij} \rangle \simeq 0$ if $|i - j| > 2$
- 2 $\langle t_{i,i+1} \rangle_{B \neq 0} > \langle t_{i,i+1} \rangle_{B=0}$



$N = 20$: $\langle t_{ij} \rangle$ for $j = 10$ and $i = j, j \pm 1, j \pm 2$ and $j \pm 3$

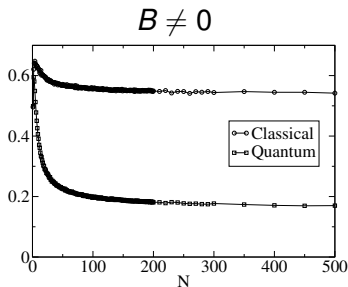
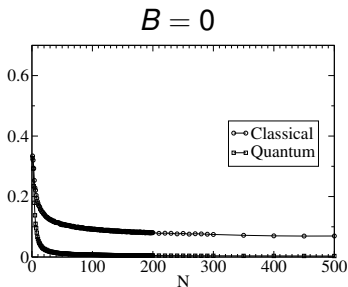
The Average Universal Conductivity $\langle \sigma(N) \rangle$



- 1 Good fit: $\langle \sigma^\infty \rangle + c/N$, where
 $\langle \sigma^\infty \rangle = \lim_{N \rightarrow \infty} \langle \sigma(N) \rangle \in (0, 1)$
- 2 Ohm and Fourier laws hold on average
- 3 Conductivity ($B \neq 0$) $>$ Conductivity ($B = 0$)

weak
 localization

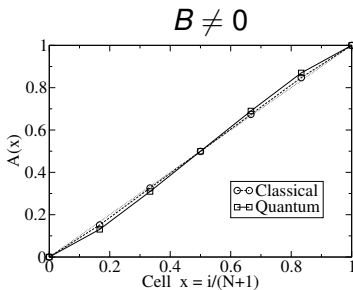
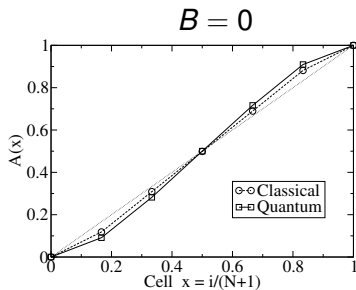
The Average Universal Conductivity $\langle \sigma(N) \rangle$



Classical conductivity $>$ Quantum conductivity weak localization

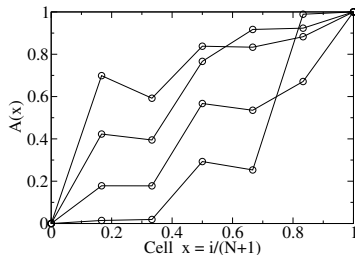
$$\langle \sigma(N) \rangle = N \left[\langle t_{LR} \rangle + \sum_{i,j=1}^N \langle t_{Lj} (\Gamma^{-1})_{ji} t_{iR} \rangle \right]$$

The Average Universal Profiles $\langle A(x) \rangle$ ($N = 5$)



- 1 Not linear/convex/concave \neq EY-model
- 2 Linear in the limit $N \rightarrow \infty$
- 3 Monotone increasing
 \rightarrow Heat also flows **locally** from Hot to Cold

Some Realizations ($N = 5$)



Local Negative Conductivity

The heat current may flow **locally** from Cold to Hot

Ref. M. Büttiker *PRB* **38** 17 (1988)

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- Onsager relations: (I_i, J_i) and $(X_i = \delta\mu_i/e, Y_i = \delta T_i/T)$

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For f MB, FD or BE

$$f(E; T_j, \mu_j) = f(E; T, \mu) - \frac{\partial f}{\partial E}(E; T, \mu) \left[\left(\frac{E - \mu}{T} \right) \delta T_j + \delta \mu_j \right]$$

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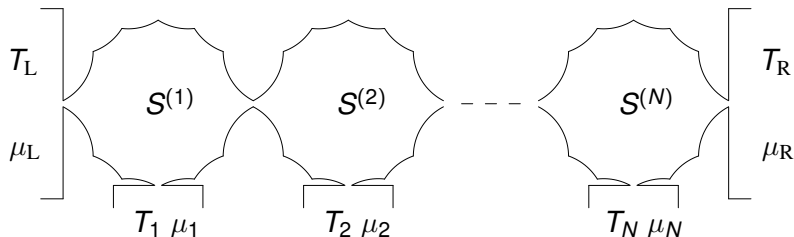
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- Classical conductivity $>$ Quantum conductivity **weak**
- Conductivity ($B \neq 0$) $>$ Conductivity ($B = 0$) **localization**

Future



- 1 Investigate the energy dependent S-matrix situation
→ 1D crystal
- 2 Analyse the effects of non-linear contributions
→ thermal rectifier
- 3 Prove the existence of the finite limit $\sigma^\infty = \lim_{N \rightarrow \infty} \sigma(N)$