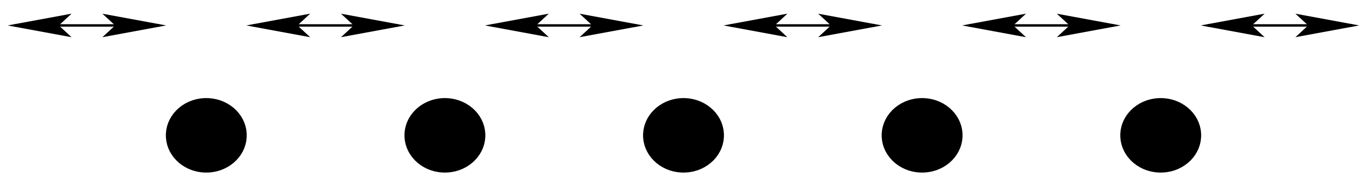
A Model of Heat Conduction

Work with Pierre Collet

+ Carlos Mejìa-Monasterio, work in progress

- Derivation of the model
- Mathematical results
- Illustrations, Approximations



The model is inspired by the scatterer model Lai-Sang Young and I developed, but in 1 spatial dimension. Novelty: a continuum of scatterers (1 per cell).

Under a Boltzmann approximation in this limit we show existence of unique non-equilibrium steady states

Discrete Space (Finite Number of Cells)

Particles of mass m and
N scatterers of mass M in a row. The scattering rules for the momenta in 1D are elastic

$$S\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} \begin{matrix} \text{(particle)} \\ \text{(scatterer)} \end{matrix}$$

$$S = \begin{pmatrix} -\sigma & 1-\sigma \\ 1+\sigma & \sigma \end{pmatrix}, \quad \sigma = (M-m)/(M+m)$$

The scatterers have finite mass M but they do NOT move
The particles move in direction of sign of momenta
They are injected (and leave) at the ends of the chain

A souvenir of the turning discs in the EY model

The "inertial" mass of the scatterers is infinite...

Assume $\sigma \in (0,1)$, that is $0 < m < M$

We arrange space in N cells (of length L each) and for each cell we have probability distributions:

$F_{L,i,in}(p)$: probability that particles with momentum $p > 0$ enter cell i from the left (per second)

...

$g_i(q)$: probability that the i-th scatterer has momentum $q \in \mathbb{R}$

$$\left(\int g = 1\right)$$

Important remark: (1D!!!) Expected time to stay in cell is $F(p)/|p|$ $\Rightarrow$ Expected number of particles in cell is infinite (when $F(0) \neq 0$)

We want to find time-stationary distributions when particles are injected from outside (out of equilibrium)

The corresponding equations are of the form (omitting index of cell)

$$g(\tilde{q}) = \frac{1}{\lambda} \int_{\mathbb{R}} dp\, g(q)\, F(p)$$

where $F = F_{L,in} + F_{R,in}$ and $\lambda = \int_{\mathbb{R}} F \approx$ particle flux

What comes out of the cell?

$$F_{L,out}(\tilde{p}) = \int_{q\ :\ \tilde{p}<0} dq\, g(q)\, F(p)$$

$$F_{R,out}(\tilde{p}) = \int_{q\ :\ \tilde{p}>0} dq\, g(q)\, F(p)$$

And the cells are coupled by $F_{L,i,in} = F_{R,i-1,out}$

Continuum Limit

$N \to \infty$, $i/N = x \in [0,1]$, scattering probability $1/N$. After some gymnastics, using things like

$$F_{L,i,in}(p) - F_{R,i,out}(p) = F_{L,i,in}(p) - F_{L,i+1,in}(p) \approx N \partial_x F(p,x)$$

one gets the Boltzmann equations

$$m\partial_t F(t,p,x) + p\partial_x F(t,p,x) = |p| \int dq \left(F(t,\tilde{p},x) g(t,\tilde{q},x) - F(t,p,x) g(t,q,x) \right)$$

$$\partial_t g(t,q,x) = \int dp \left(F(t,\tilde{p},x) g(t,\tilde{q},x) - F(t,p,x) g(t,q,x) \right)$$

The closure assumption is hidden in the independence of the F and g at different x

(It is still a good approximation)

The stationary equation $x \in [0, x_0]$

$$\partial_x F(p,x) = \mathrm{sign}(p) \int dq \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) g(q,x) \right)$$

$$0 = \int dp \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) g(q,x) \right)$$

Remark:

$$F(p,x) = \lambda e^{-\beta(p-ma)^2} \quad , \quad g(q,x) = e^{-\beta(m/M)(q-Ma)^2}$$

are solutions for all $\beta > 0$, $\lambda > 0$, and $a \in \mathbb{R}$

Want $\int dq\, g = 1$ (but not necessarily $\int dp\, F = 1$)

I omit the normalizing square roots

We would like to "bifurcate" from these equilibrium solutions by imposing $F(p, x=0)$ for $p>0$ and $F(p, x=1)$ for $p<0$

We want to impose nonequilibrium INCOMING fluxes

Main result :

This problem has solutions

Several difficulties make the result less general than Pierre and I expected (but perhaps there are better tricks)

We write $F(p,0) = \exp(-p^2) \cdot v(p)$ and define

$$C = \left\{ v \ : \ v(p) \geqslant 0, v \not\equiv 0, \int |dv(p)| + \int e^{-p^2} |v(p)| < \infty \right.$$

$$\left. \text{and } 0 < \lim_{p \to \pm\infty} Z \cdot v(p) < 1 \right\} \quad \text{with } Z = \sqrt{\pi} / \int dp\, F(p,0)$$

This is a convex Banach cone ensuring that

- v has limits (and is positive, F is a rate...)

But also

- $F(p,0)$ cannot be Gaussian, since we require

$$\lim_{p \to \pm\infty} \frac{F(p)\, e^{p^2} \int e^{-s^2} ds}{\int F(s) ds} < 1$$

- The temperatures (but not the distributions!) are the same for $p > 0$ and $p < 0$

Main Result : For any v_* in the interior of the cone C and for all initial conditions $v(\cdot, x = 0)$ near v_* the Boltzmann equation has a (unique) solution in C for $x \in [0, x_0]$ with $x_0 > 0$. The map $v(x = 0) \mapsto v(x_0)$ is a diffeomorphism

Consequence : In the image of the neighborhood, I can choose $v(p, x_0)$ for $p < 0$. In other words, within the limits of applicability of the main result, I can choose $v(p>0, x=0)$ and $v(p<0, x_0)$, that is, I can prescribe the incoming (slightly different) momentum and particle flux profiles (at the ends $x = 0$, $x = x_0$) and obtain a unique steady state

Remarks about the proof – which hopefully explains why we take these "funny" conditions on the cone

- It is not obvious that the density of $F(x,\cdot)$ remains a positive function
- The Boltzmann equation has two parts, for F with a space derivative, for g it is simpler. So we solve first for $g(x,\cdot)$, given $F(x,\cdot)$, and then integrate to find F

$$\partial_x F(p,x) = \mathrm{sign}(p) \int dq \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) \right)$$

$$g(q,x) = \frac{\int dp\, F(\tilde{p},x) g(\tilde{q},x)}{\int dp\, F(p,x)}$$

We view the second equation as a fixed point problem. Solve, and substitute into the first equation and integrate. The difficult part is the second equation. To study it fix the integral of F to 1

Since x is a spectator in the g-equation, we consider instead

$$g(q) = \int dp\, F(\tilde{p})\, g(\tilde{q})$$

N.B. $\tilde{p} = -\sigma p \quad + (1-\sigma)q$
$\tilde{q} = (1+\sigma p) + \sigma q$

This is a convolution operator, and we use spectral properties. The cone C guarantees that the r.h.s. is (for $F \in \mathrm{int}\ C$) a quasi-compact operator with isolated largest eigenvalue (equal to 1)

Conjecture: the essential spectral radius ends at the larger of the 2 limits of $v(p)$:
the numerical spectrum is $\{\sigma^n\}_{n=0}^{\infty}$

Let $\mu \equiv \frac{m}{M} = \frac{1-\sigma}{1+\sigma}$ and $g(q) = \exp(-\mu q^2) u(q)$ and define

$$\|u\|_1 = \int dq\, e^{-\mu q^2} |u(q)|$$

and

$$\|u\|_2 = \int dq\, e^{-\mu q^2} |u(q)| + \int |du(q)| \ .$$

The main estimate is then for any $v \in C$:

There exist a $\zeta < 1$ and an $R > 0$ (both depend on v continuously) such that the convolution operator $K_v \Leftrightarrow \int dp F(\tilde{p}) g(\tilde{q})$ satisfies for any $\|u\|_2 < \infty$ the bound

$$\int |dK_v(u)| \leqslant \zeta \int |du| + R\|u\|_1$$

Souvenirs de Lasota-Yorke détails pour amateurs en privé...

Why can't we have $\beta \neq \beta'$?

$$F(p,0) = \begin{cases} \exp(-\beta p^2), & p > 0 \\ \exp(-\beta' p^2), & p < 0 \end{cases}$$

The problem is that convolution mixes contributions from positive and negative p. The reflection by the scatterer exchanges temperature information between the positive to the negative momentum side

This makes us lose compactness, and we dont know how to show existence and uniqueness of g without some information. Numerics shows it is much better...

The discontinuity of F at $p = 0$ is a realistic phenomenon and is the reason why we consider the variation norm $\int |dv(p)|$ instead of $\int dp\, |v'(p)|$. With such norms the compactnes of convolution is well known (one gains a derivative) and in fact the probability densities are smooth in p and q except at $p = 0$

Numerical study

Compare the Boltzmann model to the discrete model from which it is derived

- One does not need the cone C
- The role of x_0 should become clear

Reconsider

$$\partial_x F(p,x) = \gamma\, \mathrm{sign}(p) \int dq \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) g(q,x) \right)$$

$$0 = \int dp \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) g(q,x) \right)$$

This Boltzmann limit is obtained by assuming in the particle model that

particles interact (in 1 cell) with probability γ/N and cross the cell without collision with probability $1-\gamma/N$

As $N \to \infty$ the cross-section of the scatterers is assumed to be γ/N.

Boltzmann simulations

$$\partial_x F(p,x) = \mathrm{sign}(p) \int dq \left(F(\tilde{p},x) g(\tilde{q},x) - F(p,x) \right)$$

$$g(q,x) = \frac{\int dp\, F(\tilde{p},x) g(\tilde{q},x)}{\int dp\, F(p,x)}$$

We discretize the space of p and q and integrate from $x = 0$ to $x = 1$. The second equation is an eigenvalue problem with unknown eigenfunction $g = g_{F(\cdot,x)}$

Substitute the g:

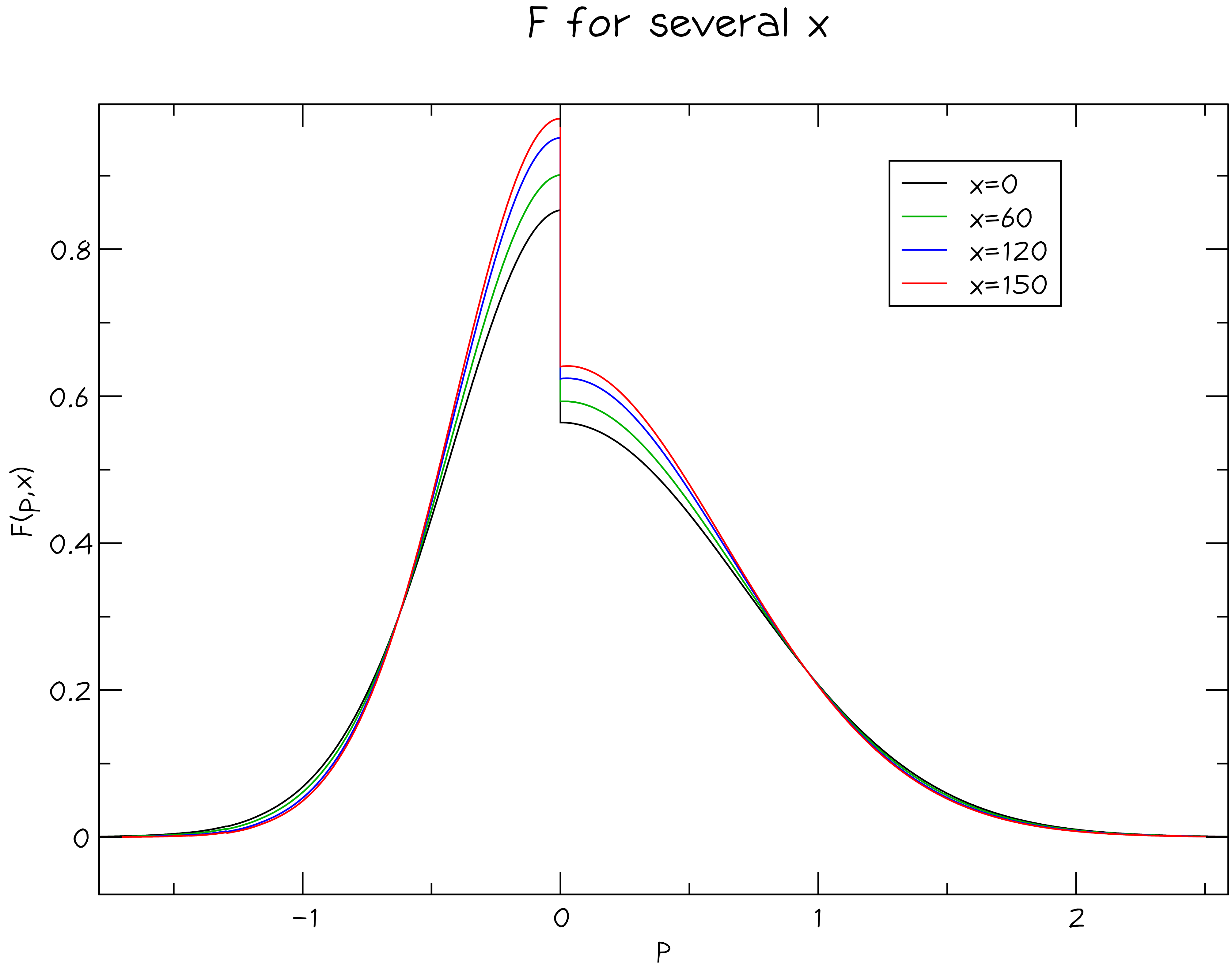
$$\partial_x F(p,x) = \mathrm{sign}(p) \int dq \left(F(\tilde{p},x) g_{F(\cdot,x)}(\tilde{q},x) - F(p,x) \right)$$

Take as initial condition

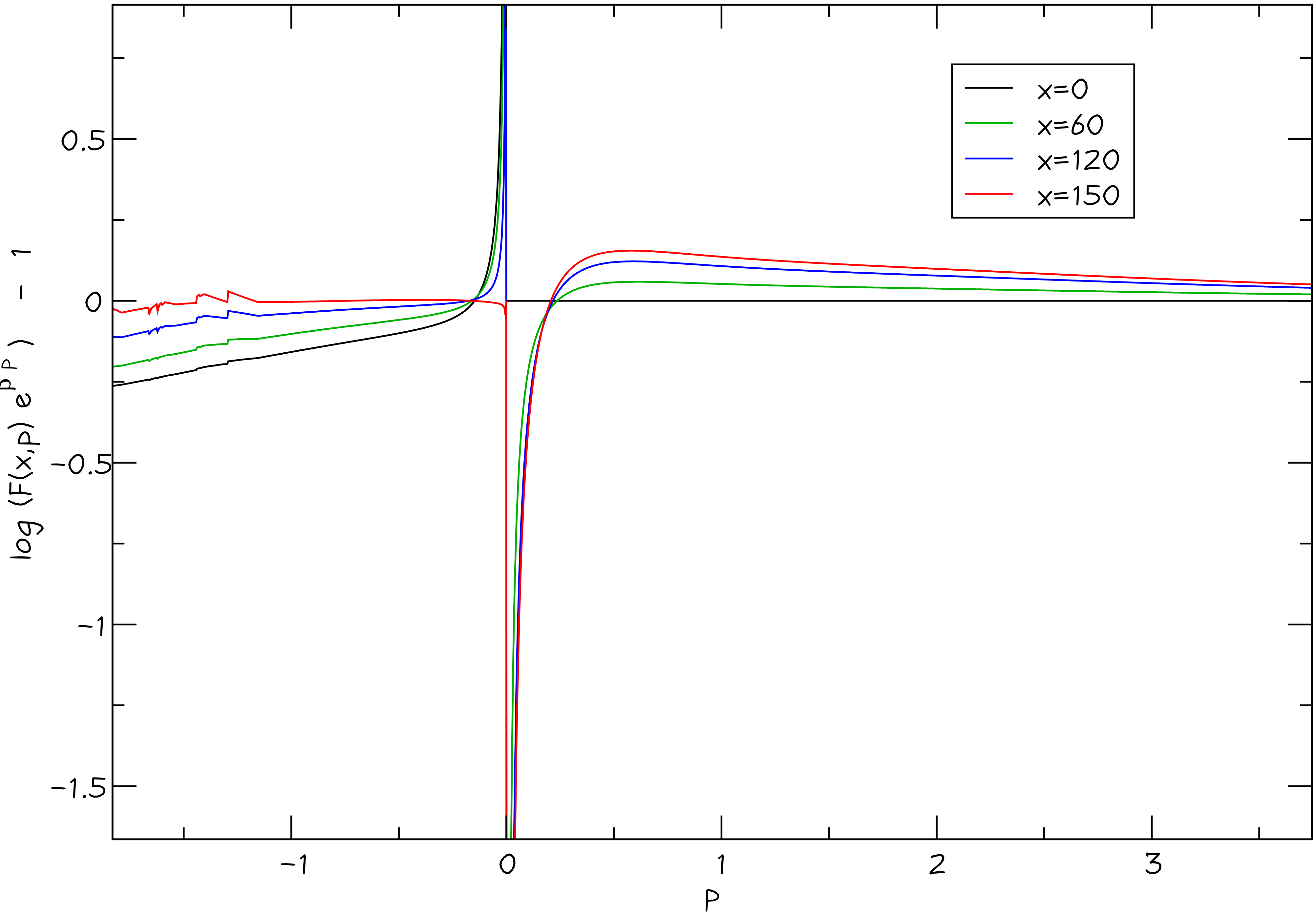
$$F_0(p,0) = \begin{cases} \lambda \exp(-\beta p^2), & p > 0 \\ \lambda' \exp(-\beta' p^2), & p < 0 \end{cases}$$

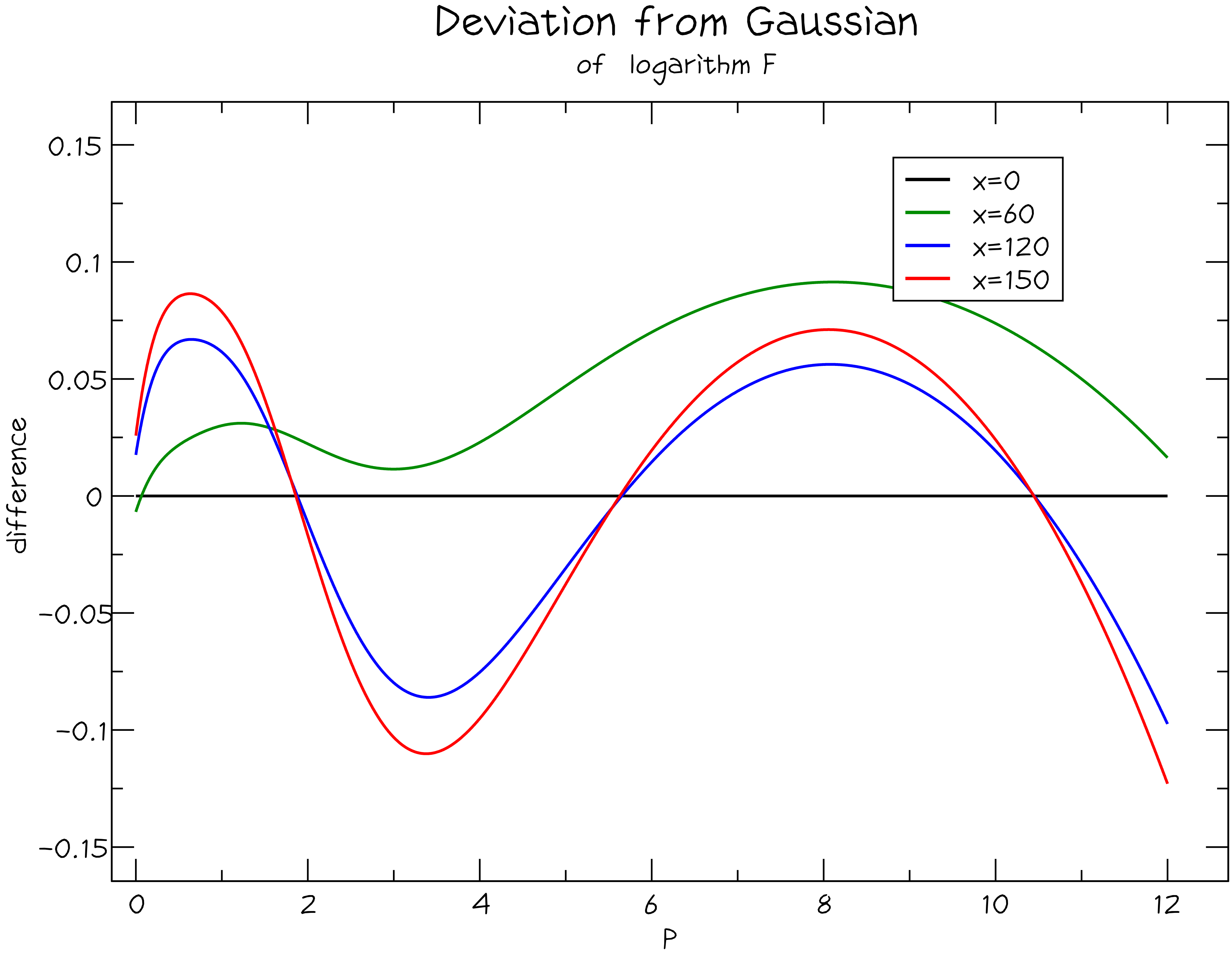
and see if $F(p, x = 1)$ for $p < 0$ is the desired incoming Gaussian $\lambda' \exp(-\beta' p^2)$. Iteratively correct $F_0 \to F_1 \to$

poor man's inverting the diffeomorphism of the Theorem

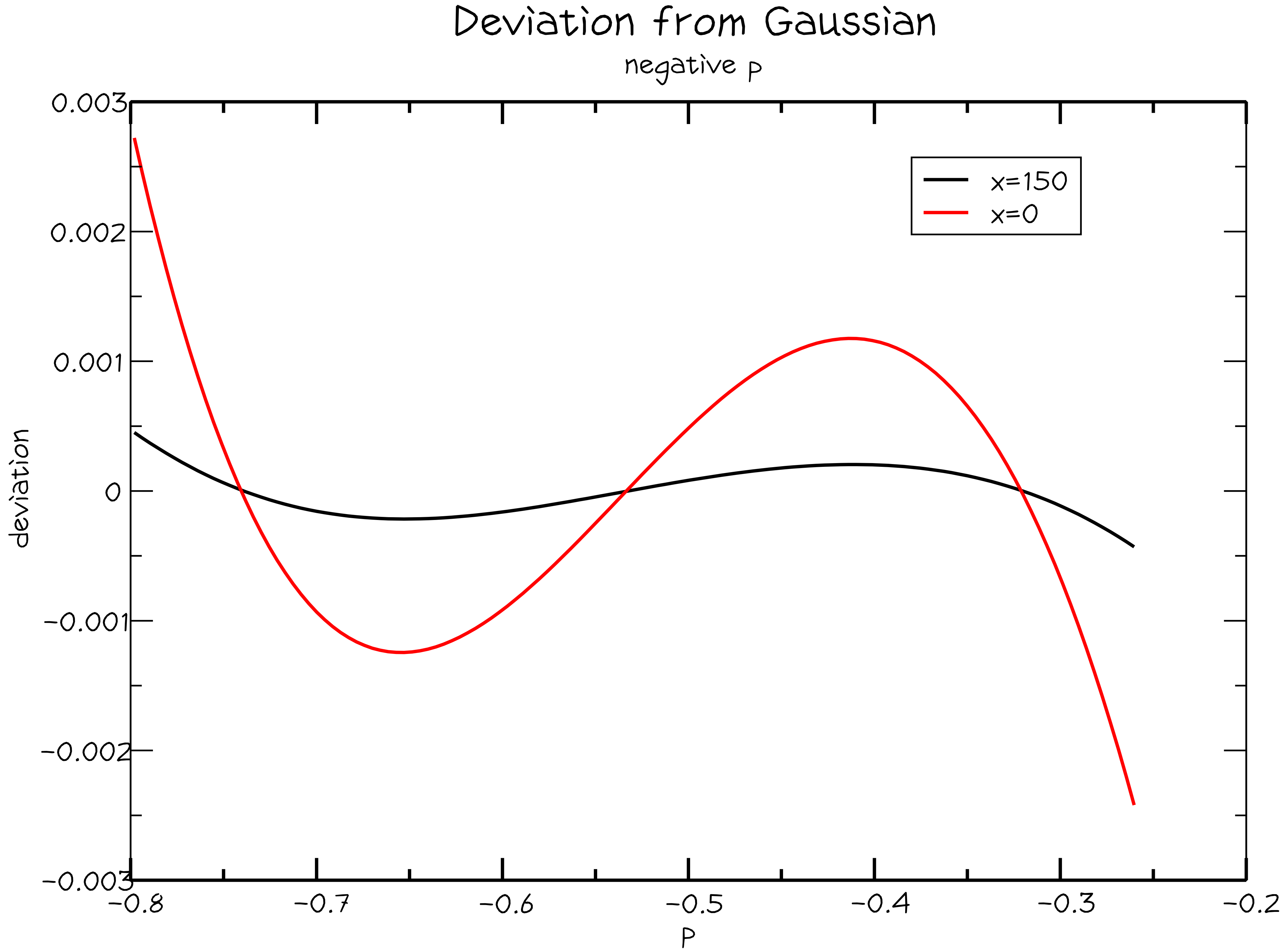

F for several x
F(p,x)
p
x=0
x=60
x=120
x=150
0.8
0.6
0.4
0.2
0
−1
0
1
2

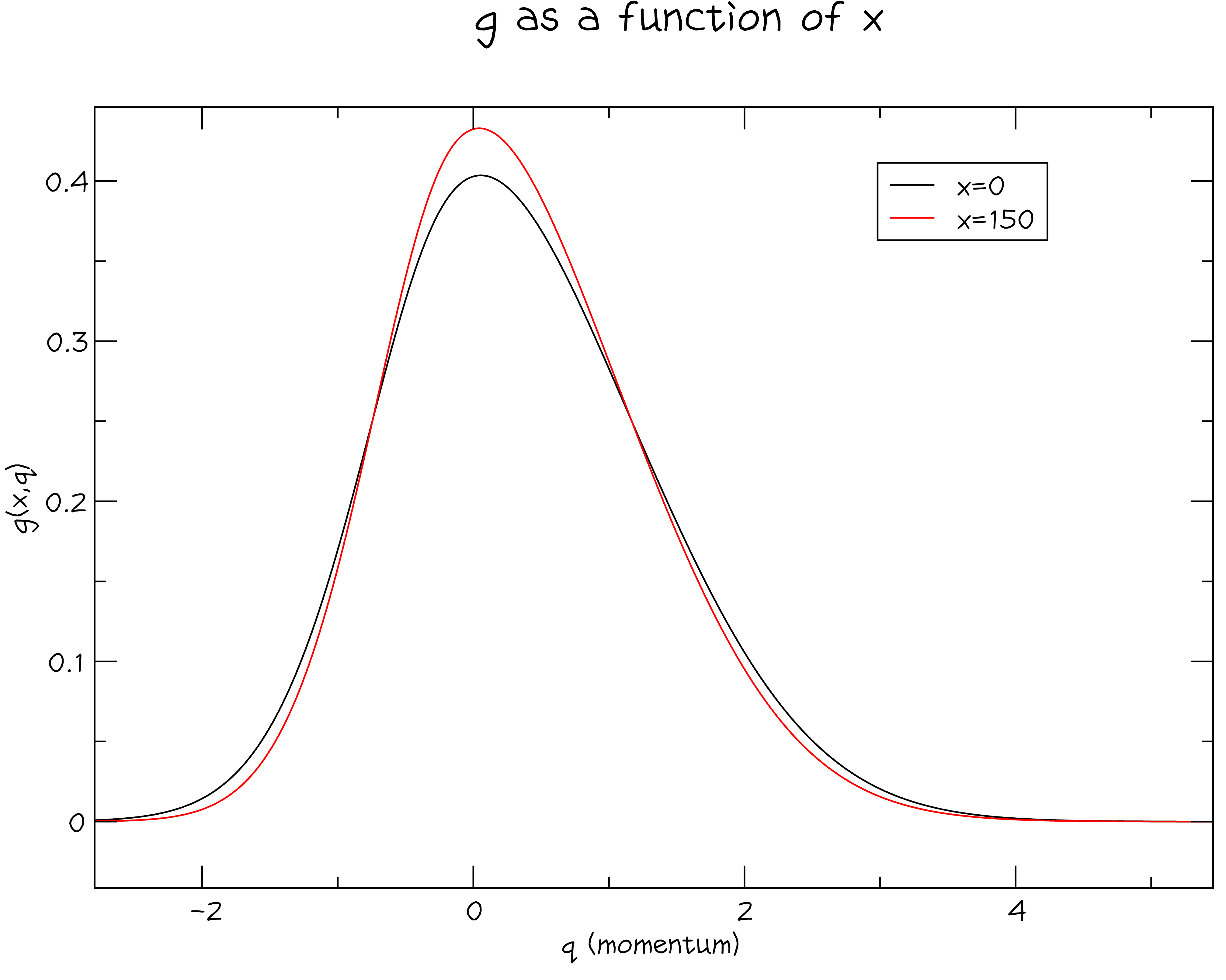
deviation from Gaussian





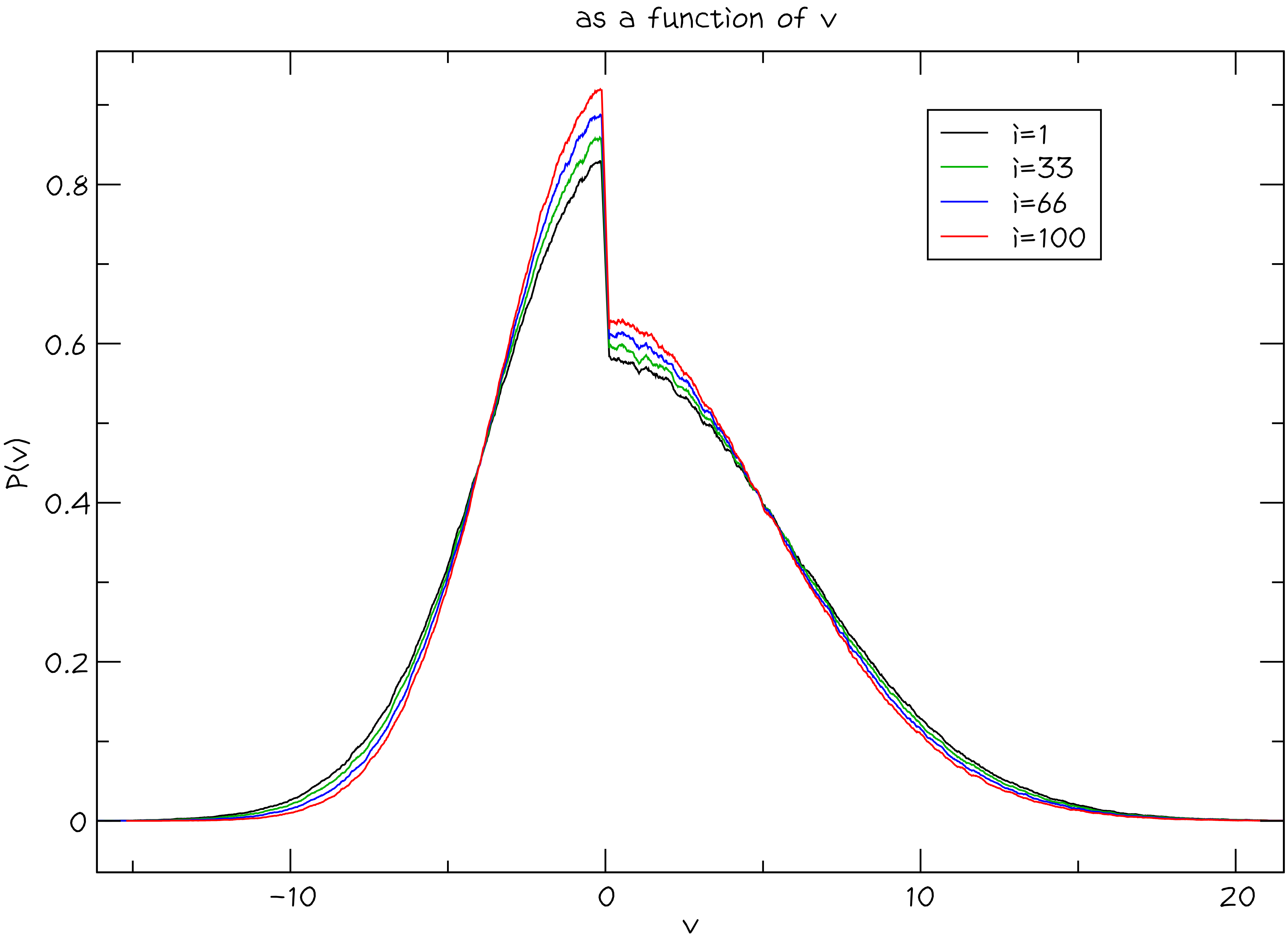
Deviation from Gaussian
of logarithm F
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
difference
0
2
4
6
8
10
12
P
x=0
x=60
x=120
x=150


Deviation from Gaussian
negative p
0.003
0.002
0.001
0
−0.001
−0.002
−0.003
deviation
−0.8
−0.7
−0.6
−0.5
−0.4
−0.3
−0.2
p
x=150
x=0

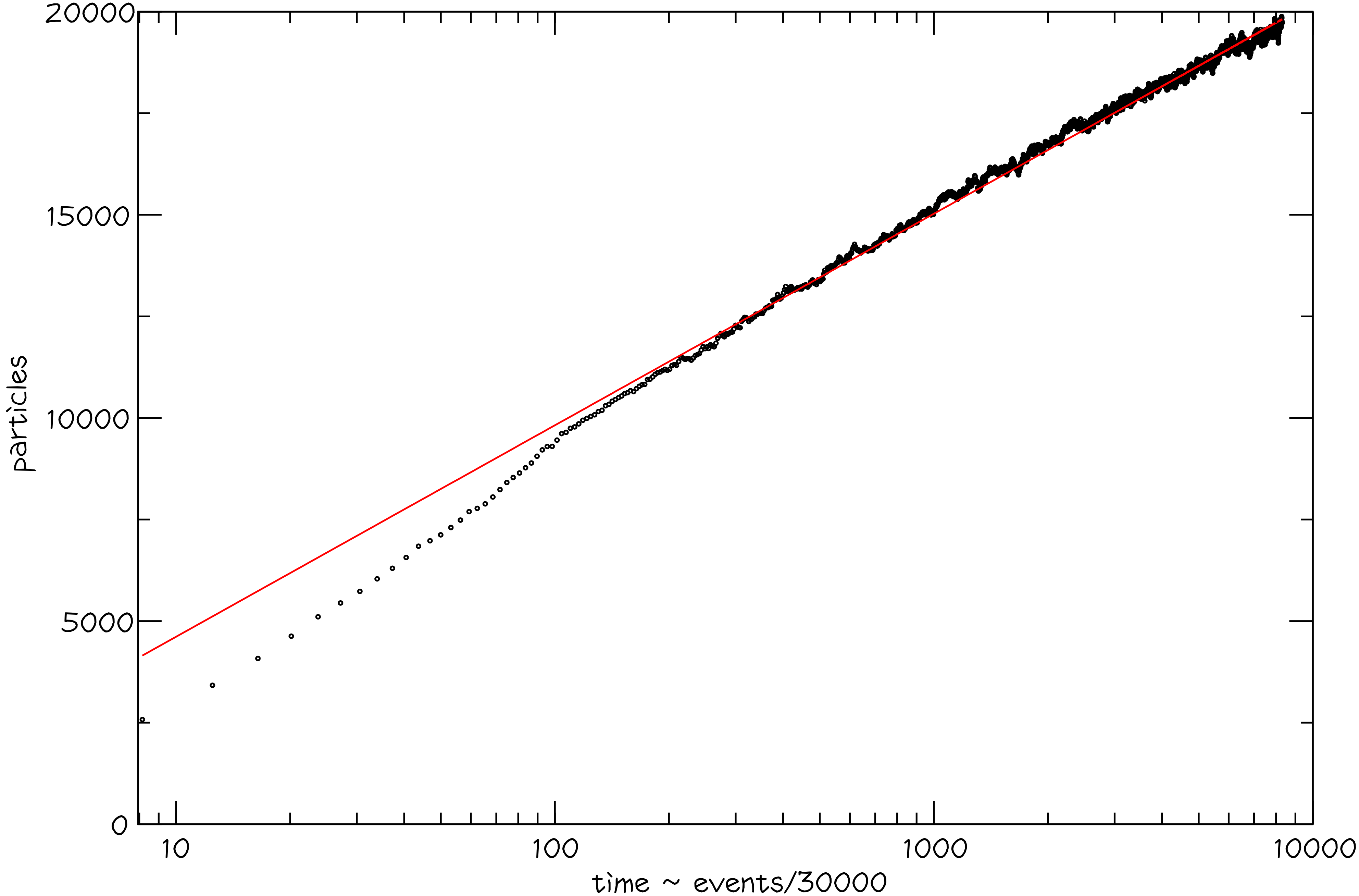


g as a function of x
x=0
x=150
0.4
0.3
0.2
0.1
0
g(x,q)
−2
0
2
4
q (momentum)

Particle simulations: $T_L = 3T_R$, $j_L = j_R$ Will take $\gamma = 1$.
Distribution of velocity



of particles as function of time



This is a typical problem of non-normalizable measures. The number of particles is proportional to $F(p)/|p|$ which is non-integrable in 1 dimension.

Similar to tangents in 1-d maps: Collet-Ferrero Annales de l'Institut Henri Poincaré (A) Physique théorique, 52 (1990), p. 283-301

Conclusion

Back to Boltzmann: We found the solution by iterating the initial distribution until the other end was what we wanted. $F_0 \to F_1 \to \cdots$

Interesting problem: What if in this problem $F_n(p, 0)$ ceases to be positive after n iterations when $p < 0$?

Cannot extract arbitrary energy profiles for $p < 0$ at $x = 0$ by injecting something at $x = 1$. (This is why we had x_0 in the theorem)

QUESTION: What ARE the possible exit distributions?