Universität Wien
Prof. Dr. D. Burde

Lie algebras and representation theory
— Exercises —

Exercises SS 2021

Exercise 1. Let V be the real vector space $\mathbb{R}^3$. Show that the cross product defines a Lie bracket on V with Lie algebra $\mathfrak{so}_3(\mathbb{R})$.

Exercise 2. Let $\mathfrak{g}$ be a finite-dimensional non-abelian Lie algebra over a field K. Show that $\dim Z(\mathfrak{g}) \leq \dim \mathfrak{g} - 2$.

Exercise 3. Let $\mathfrak{g}$ be the complex Lie algebra with basis (x, y, z) and defining Lie brackets $[x, y] = z$ and $[x, z] = y$. Decide for each statement, whether it is true or not and give a proof.

(1) $\mathfrak{g}$ is reductive.

(2) $\mathfrak{g}$ is solvable.

(3) $\mathfrak{g}$ is nilpotent.

(4) There is a surjective homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$.

Exercise 4. Show that the semidirect sum of two solvable Lie algebras is solvable. Is it true that the semidirect sum of two nilpotent Lie algebras is nilpotent?

Exercise 5. Let $\mathfrak{g}$ be a solvable Lie algebra of characteristic zero and $\mathrm{nil}(\mathfrak{g})$ be the nilradical of $\mathfrak{g}$. Show that $D(\mathfrak{g}) \subseteq \mathrm{nil}(\mathfrak{g})$ for all derivations $D \in \mathrm{Der}(\mathfrak{g})$.

Exercise 6. Denote by $\mathfrak{h}_5$ the Heisenberg Lie algebra of dimension 5, over a field K of characteristic zero, with basis (x_1, x_2, y_1, y_2, z) and defining Lie brackets $[x_1, y_1] = [x_2, y_2] = z$. Construct a faithful 4-dimensional representation $\varphi\colon \mathfrak{h}_5 \to \mathfrak{gl}_4(K)$.

Exercise 7 - extra. Give an example of a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero, such that not every element in $[\mathfrak{g}, \mathfrak{g}]$ can be written as a commutator $[x, y]$.

Exercise 8. Let $\mathfrak{g}$ be a Lie algebra and $V_1, \ldots, V_n$ be $\mathfrak{g}$-modules. Show that the tensor product $V_1 \otimes \cdots \otimes V_n$ becomes a $\mathfrak{g}$-module by the rule

$$x.(v_1 \otimes \cdots \otimes v_n) = (x.v_1 \otimes \cdots \otimes v_n) + (v_1 \otimes x.v_2 \otimes \cdots \otimes v_n) + \cdots + (v_1 \otimes \cdots \otimes x.v_n)$$

for $x \in \mathfrak{g}$ and $v_i \in V_i$.

Exercise 9. Denote by $\mathbb{C}^2$ the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension 2. Show that the $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the direct sum of two simple $\mathfrak{sl}_2(\mathbb{C})$-modules and determine these simple modules.

Exercise 10. Determine the Lie algebra of derivations of $\mathfrak{gl}_n(\mathbb{C})$ for $n \geq 2$ and show that there is no invertibe derivation.

Exercise 11. Let $\mathfrak{g}$ be a Lie algebra of characteristic zero, $\mathrm{rad}(\mathfrak{g})$ its solvable radical and $\mathrm{nil}(\mathfrak{g})$ be its nilradical. Show that $\mathrm{nil}(\mathrm{rad}(\mathfrak{g})) = \mathrm{nil}(\mathfrak{g})$.

Exercise 12. Let $\mathfrak{g}$ be a n-dimensional solvable Lie algebra over an algebraically closed field of characteristic zero. Show that $\mathfrak{g}$ has an ideal I_i of dimension i for every $0 \leq i \leq n$. Find such ideals for the Lie algebra of Exercise 3.

Exercise 13. Determine the Killing form $\kappa(x, y)$ for the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ and conclude that both Lie algebras are simple but not isomorphic.

Exercise 14 - extra. Show that the determinant of the Killing form for $\mathfrak{sl}_n(K)$ with respect to the standard basis is given by

$$\det(\kappa) = \left((-1)^{\frac{n(n-1)}{2}}\right) 2^{n^2-1} n^{n^2}.$$

Due: June 30, 2021