Cohomology of Groups and Algebras

Dietrich Burde

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Contents

CHAPTER 1

Introduction

Homology and cohomology has its origins in topology, starting with the work of Riemann (1857), Betti (1871) and Poincaré (1895) on "homology numbers" of manifolds. Although Emmy Noether observed in 1925 that homology was an abelian group rather than just Betti numbers, homology remained a part of the realm of topology until about 1945. During the period of 1940 – 1955 came the rise of algebraic methods. The homology and cohomology of several algebraic systems were defined and explored: Tor and Ext for abelian groups, homology and cohomology of groups and Lie algebras, the cohomology of associative algebras, sheaves, sheaf cohomology and spectral sequences. At this point the book of Cartan and Eilenberg (1956) crystallized and redirected the field completely. Their systematic use of derived functors, defined by projective and injective resolutions of modules, united all the previously disparate homology theories. Several new fields grew out of this: homological algebra, K-theory, Galois theory, étale cohomology of schemes and so on. Much could be said also on newer developments in homological algebra.

Concerning group cohomology, the low dimensional cohomology of a group G was already classically studied in other guises, long before the formulation of group cohomology in 1943 – 1945 by Eilenberg and MacLane. For example, classical objects were

$$H^0(G,A) = A^G,\ H^1(G,\mathbb{Z}) = G/[G,G]$$

and for G finite, the character group

$$H^2(G,\mathbb{Z}) = H^1(G,\mathbb{C}^\times) = \mathrm{Hom}(G,\mathbb{C}^\times)$$

Also the group $H^1(G,A)$ of crossed homomorphisms of G into a G-module A is classical as well: Hilbert's Theorem 90 from 1897 is actually the calculation that $H^1(G,L^\times) = 0$ when G is the Galois group of a cyclic field extension L/K. One should also mention the group $H^2(G,A)$ which classifies extensions over G with normal abelian subgroup A via factor sets. The idea of factor sets appeared already in Hölders paper in 1893 and again in Schur's paper in 1904 on projective representations $G \to PGL_n(\mathbb{C})$. Schreier's paper in 1926 was the first systematic treatment of factor sets, without the assumption that A is abelian.

Lie algebra cohomology was invented by Elie Cartan, Claude Chevalley und Samuel Eilenberg around 1950 to compute the de Rham cohomology of a compact Lie group. Cartan had shown that the computation of the cohomology of Lie groups can be reduced to the cohomology of compact Lie groups. Chevalley and Eilenberg defined in [**7**] first the Lie algebra cohomology $H^n(\mathfrak{g}, \mathbb{R})$ with the trivial $\mathfrak{g}$-module $\mathbb{R}$, by transfering the de Rham cohomology $H^n_{dR}(G, \mathbb{R})$ of a connected compact Lie group to its Lie algebra. This yields an isomorphism $H^n_{dR}(G, \mathbb{R}) \cong H^n(\mathfrak{g}, \mathbb{R})$, where $\mathfrak{g}$ is the Lie algebra of G. To study the cohomology $H^n(G/H, \mathbb{R})$ of homogeneous spaces G/H of G, Chevalley und Eilenberg defined the Lie algebra cohomology $H^n(\mathfrak{g}, M)$ for a general $\mathfrak{g}$-Modul M. Furthermore the article [**7**] already contains the interpretation of $H^2(\mathfrak{g}, M)$ by Lie algebra extensions of $\mathfrak{g}$ by M, as well as the Whitehead Lemmas in the form $H^1(\mathfrak{g}, M) = H^2(\mathfrak{g}, M) = 0$ for finite-dimensional semisimple Lie algebras over a field of characteristic zero and finite-dimensional $\mathfrak{g}$-modules M.