Universität Wien
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Group Theory
— Exercises —

Exercises SS 2020

Exercise 1. Let K be a field of characteristic different from 2. Show that the set of matrices

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in K$ forms a non-abelian subgroup of $GL_3(K)$. Find an explicit formula for all powers A^n, with $n \in \mathbb{Z}$.

Exercise 2. Show that $\mathrm{Aut}(\mathrm{S}_3) \cong \mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) \cong \mathrm{S}_3$.

Exercise 3. Determine (without proof) the number n of different groups of order 1536. Show that n is not the sum of three integer cubes, but find integers x, y, z with $1536 = x^3 + y^3 + z^3$.

Exercise 4. Determine the number of elements in the group $GL_n(\mathbb{F}_q)$.

Exercise 5. Show that every subgroup of the quaternion group Q_8 is normal. Is this true for the dihedral group D_4?

Exercise 6. Show that the groups $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Q}$ cannot be isomorphic.

Exercise 7 - extra. Determine the order of the group $G = SL_2(\mathbb{F}_3)$, and show that G has exactly one 2-Sylow subgroup.

Exercise 8. Show that the symmetric group S_n has a subgroup of index n for all $n \geq 1$.

Exercise 9. Show that every finite group can be embedded into a finite simple group.

Exercise 10. Show that there is no simple group of order 312.

Exercise 11. Show that the quaternion group Q_8 cannot be written as a semidirect product of two non-trivial subgroups.

Exercise 12. Classify all groups which are a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}$. Show that there is one abelian group and one non-abelian group.

Exercise 13. Determine all composition series for the quaternion group Q_8.

Exercise 14 - extra. Let d be a squarefree integer and $a, b \in \mathbb{Z}$. Show that the field extension $\mathbb{Q}(\sqrt{a + b\sqrt{d}}) \mid \mathbb{Q}$ is a Galois extension if and only if $\frac{a-b\sqrt{d}}{a+b\sqrt{d}}$ is a square in $\mathbb{Q}(\sqrt{d})$ and determine its Galois group in this case.

Exercise 15. Let $G = S_7$ and $x = (1234567)$, $y = (265734)$ in G with $H := \langle x, y \rangle$. Find a composition series for H and determine its quotients.

Exercise 16. Let G be a nilpotent group and H a non-trivial normal subgroup of G. Show that $H \cap Z(G)$ is non-trivial.

Exercise 17. Show that the infinite dihedral group D_∞ is solvable.

Exercise 18. Show that D_n is solvable for all $n \in \mathbb{N}$, and nilpotent if and only if n is a power of two.

Exercise 19. Let G be a non-abelian group of order pq, where p and q are distinct primes. Show that G is solvable, but not nilpotent.

Exercise 20. Show that the projective special linear group $PSL_2(\mathbb{F}_q)$ is isomorphic to a subgroup of S_{q+1} and determine its order.

Exercise 21 - extra. Let G be a finite solvable group, all of whose Sylow subgroups are abelian. Prove that $Z(G) \cap G' = 1$.

Exercise 22. Let G be a nilpotent group of cube-free order n. Show that G is abelian.

Exercise 23. Let G be a finite group. Show that G is nilpotent if and only if $xy = yx$ for all $x, y \in G$ having relatively prime orders.

Exercise 24. Show that the group $PSL_2(\mathbb{F}_3)$ is isomorphic to A_4 and that $PSL_2(\mathbb{F}_5)$ is isomorphic to A_5. Is $PSL_2(\mathbb{F}_7)$ also isomorphic to an alternating group?

Exercise 25. Show that the presentation

$$\langle a, b, c \mid a^3 = b^3 = c^4 = acac^{-1} = aba^{-1}bc^{-1}b^{-1} = e \rangle$$

defines the trivial group.

Exercise 26. Show that the infinite dihedral group $D_\infty = C_2 * C_2$ has derived length 2 by computing D'_∞ and D''_∞.

Exercise 27. Prove that every free group of rank greater than one has an infinite number of free generating sets.

Exercise 28 - extra. Show that the presentation

$$\langle a, b \mid ab^2a^{-1}b^{-3} = ba^2b^{-1}a^{-3} = e \rangle$$

defines the trivial group.

Exercise 29. Let G be a group and denote by $\mathrm{Ab}(G) = G/[G, G]$ its abelianization. Show that $\mathrm{Ab}(G * H) \cong \mathrm{Ab}(G) \oplus \mathrm{Ab}(H)$.

Exercise 30. Let F_n denote the free group of rank n. Use exercise 26 to show that

$$F_n/[F_n, F_n] \cong \mathbb{Z}^n.$$

Exercise 31. Let $G = H * K$ with non-trivial subgroups H and K. Show that the center of G is trivial.

Exercise 32. Let $G = H *_M K$ with different subgroups H, K and M. Show that the center of G is contained in M.

Exercise 33. Let G, H and K be groups such that $G * H = G * K$. Does it follow that $H \cong K$? Give a proof or give a counterexample.

Exercise 34. Let A and B be matrices from $SL_2(\mathbb{Z})$ given by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}.$$

Show that the semigroup generated by A and B under multiplication is not free.

Exercise 35- extra. Let $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} G \to 1$ be a group extension of A by G and assume that A is abelian.

(a) For $x \in G$ choose an $e \in E$ such that $\beta(e) = x$. Then set $x \cdot a = eae^{-1}$ for $a \in A$. Show that this is a well-defined action of G on A so that A becomes a G-module.

(b) Show that equivalent extensions of A by G give rise to the same action.

Exercise 36. Let $n \geq 1$ be an integer. Give group homomorphisms α and β such that

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/n\mathbb{Z} \to 0$$

becomes a short exact sequence of abelian groups. For which n is this sequence split ?

Exercise 37. Let $G = A_5$ be the alternating group on five letters and $A = \mathbb{Z}/7\mathbb{Z}$ be a trivial A_5-module. Compute $H^n(G, A)$ for $n = 0, 1, 2$.

Exercise 38. Let G be a finite group and $\mathbb{Q}$ be a G-module. Show that $H^n(G, \mathbb{Q}) = 0$ for all $n \geq 1$.

Exercise 39. Let G be a group and A be a G-module. Fix a τ in the center of G. Let $\varphi\colon Z^1(G, A) \to C^1(G, A)$ be the map $f \mapsto \varphi(f)$ where $\varphi(f)(\sigma) = \tau f(\sigma) - f(\sigma)$ for $\sigma \in G$. Show that $H^1(G, A)$ is annihilated by φ, i.e., $\varphi(Z^1(G, A)) \subseteq B^1(G, A)$.

Exercise 40. Determine the group $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ by computing the 2-cocycles and 2-coboundaries, where $\mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ acts nontrivially on $\mathbb{Z}$.

Exercise 41. Let $\mathbb{Z}$ be a trivial $\mathbb{Q}$-module. Compute $H^n(\mathbb{Q}, \mathbb{Z})$ for $n = 0, 1$.

Exercise 42 - extra. Understand the notation of the following lemma and give a proof.

Shapiro's Lemma: Let H be a subgroup of a group G. For any H-module M, there is a caconical isomorphism

$$H^n(G, \operatorname{Ind}_H^G(M)) \xrightarrow{\cong} H^n(H, M).$$

Due: June 30, 2020