Universität Wien
Prof. Dr. D. Burde

Cohomology of Groups and Algebras
— Exercises —

Exercises WS 2022/23

Exercise 1. Show that the quaternion group Q_8 cannot be written as a semidirect product of two non-trivial subgroups.

Exercise 2. Let

$$1 \to G_1 \to G_2 \to \cdots \to G_n \to 1$$

be an exact sequence of finite groups. Show that

$$\prod_{i=1}^{n} |G_i|^{(-1)^i} = 1.$$

Exercise 3. Show that an extension of a nilpotent group by a nilpotent group need not be nilpotent.

Exercise 4. Classify all groups which are a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}$.

Exercise 5. Let $n \geq 1$ be an integer. Find group homomorphisms α and β such that

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/n\mathbb{Z} \to 0$$

becomes a short exact sequence of abelian groups. For which n is this sequence split ?

Exercise 6. Consider the following commutative diagram of groups and homomorphisms with exact rows.

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5
\end{array}$$

Prove the following claims.

(a) If f_2, f_4 are onto and f_5 is one-to-one, then f_3 is onto.

(b) If f_2, f_4 are one-to-one and f_1 is onto, then f_3 is one-to-one.

(c) If f_1, f_2 and f_4, f_5 are isomorphisms, so is f_3.

Exercise 7. Let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \subseteq SL_3(\mathbb{Z})$$

be the integer Heisenberg group. Show that H is a split extension of $\mathbb{Z}^2$ by $\mathbb{Z}$, i.e., there is a split short exact sequence

$$1 \to \mathbb{Z}^2 \xrightarrow{i} H \xrightarrow{\pi} \mathbb{Z} \to 1.$$

Exercise 8. Let H be the integer Heisenberg group and consider the maps $i\colon \mathbb{Z} \to H$, $c \mapsto \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\pi\colon H \to \mathbb{Z}^2$, $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, b)$. Show that we obtain a short exact sequence

$$1 \to \mathbb{Z} \xrightarrow{i} H \xrightarrow{\pi} \mathbb{Z}^2 \to 1,$$

which does not split.

Exercise 9. Let p be a prime and $\mathbb{Z}_{(p)} = \{\frac{n}{m} \in \mathbb{Q} \mid p \nmid m\}$ the ring of p-local numbers. Let $\mathbb{Z}/p^\infty = \cup_{i \geq 0} \mathbb{Z}/p^i$ with the uniquely given embeddings $\mathbb{Z}/p_i \hookrightarrow \mathbb{Z}/p^{i+1}$. Show that there is a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty \to 0$$

which does not split.

Exercise 10. Let $\mathbb{Z}[G]$ be the integral group ring of a finite group G and denote by

$$U(\mathbb{Z}[G]) = \{\alpha \in \mathbb{Z}[G] \mid \alpha\beta = \beta\alpha = 1 \text{ for some } \beta \in \mathbb{Z}[G]\}$$

the unit group of $\mathbb{Z}[G]$. Let $a, b \in G$ with $\mathrm{ord}(a) = n$. Show that the following elements are always units:

$$1 + (1-a)b(1+a+\cdots+a^{n-1}) \text{ and } 1 + (1+a+\cdots+a^{n-1})b(1-a).$$

Exercise 11. Let $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} G \to 1$ be a group extension of A by G and assume that A is abelian.

(a) For $x \in G$ choose an $e \in E$ such that $\beta(e) = x$. Then set $x \cdot a = eae^{-1}$ for $a \in A$. Show that this is a well-defined action of G on A so that A becomes a G-module.

(b) Show that equivalent extensions of A by G give rise to the same action.

Exercise 12. Let $G = A_5$ be the alternating group on five letters and $A = \mathbb{Z}/7\mathbb{Z}$ be a trivial A_5-module. Compute $H^n(G, A)$ for $n = 0, 1, 2$.

Exercise 13. Let G be a finite group and $\mathbb{Q}$ be a G-module. Show that $H^n(G, \mathbb{Q}) = 0$ for all $n \geq 1$.

Exercise 14. Let G be a group and A be a G-module. Fix a τ in the center of G. Let $\varphi\colon Z^1(G, A) \to C^1(G, A)$ be the map $f \mapsto \varphi(f)$ where $\varphi(f)(\sigma) = \tau f(\sigma) - f(\sigma)$ for $\sigma \in G$. Show that $H^1(G, A)$ is annihilated by φ, i.e., $\varphi(Z^1(G, A)) \subseteq B^1(G, A)$.

Exercise 15. Determine the group $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ by computing the 2-cocycles and 2-coboundaries, where $\mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ acts nontrivially on $\mathbb{Z}$.

Exercise 16. Let $\mathbb{Z}$ be a trivial $\mathbb{Q}$-module. Compute $H^n(\mathbb{Q}, \mathbb{Z})$ for $n = 0, 1$.

Exercise 17. Verify that the functor $\mathrm{Hom}(-, \mathbb{Z})$ is not exact by applying it to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Exercise 18. Show that the category of groups is not abelian.

Exercise 19. Show that the category of free abelian groups is not abelian.

Exercise 20. Show that the category of finite abelian groups does not have enough injective objects and not enough projective objects.

Exercise 21. Show that the category of cyclic groups does not have all products and hence is not an abelian category.

Exercise 22. Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over a field K of characteristic zero. Show that $H^1(\mathfrak{g}, \mathfrak{g}) = 0$.

Exercise 23. Sei $\mathfrak{g} = \mathfrak{sl}_2(K)$ over a field K of of arbitrary characteristic. Compute the dimension of $H^1(\mathfrak{g}, \mathfrak{g})$.

Exercise 24. Let $\mathfrak{n}_4(K)$ be the Lie algebra with basis $(e_1, \dots, e_4)$ and Lie bracktes $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$. Compute $H^2(\mathfrak{n}_4(K), K)$.

Exercise 25. Show that $H^2(\mathfrak{g}, K)$ may be regarded as a subspace of $H^1(\mathfrak{g}, \mathfrak{g}^*)$, where $\mathfrak{g}^*$ denotes the coadjoint module.

Exercise 26. Let $\mathfrak{g}$ be a Lie algebra over K having no nonzero invariant bilinear form. Show that $H^2(\mathfrak{g}, K) \cong H^1(\mathfrak{g}, \mathfrak{g}^*)$.

Exercise 27. Let $\mathfrak{g}$ be a semisimple Lie algebra over a field K of characteristic zero. Show that $H^3(\mathfrak{g}, K) \neq 0$.

Exercise 28. Use Lie algebra cohomology to classify the extensions of a 2-dimensional abelian Lie algebra by a 1-dimensional Lie algebra.

Exercise 29. Show that the Heisenberg Lie algebra $\mathfrak{n}_3(K)$ is a non-split extension of K^2 by K.

Exercise 30. Use Whitehead's second lemma to give a proof of Levi's theorem: if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero, then there is a semisimple subalgebra $\mathfrak{s}$ such that

$$\mathfrak{g} \cong \operatorname{rad}(\mathfrak{g}) \rtimes \mathfrak{s}.$$

Exercise 31. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K. Show that the Euler characteristic of $\mathfrak{g}$ is zero, i.e.,

$$\chi(\mathfrak{g}) = \sum_{i=0}^{n} (-1)^i \dim H^i(\mathfrak{g}, K) = 0.$$

Exercise 32. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension $n \geq 2$ over a field K and

$$\sigma(\mathfrak{g}) = \sum_{i=0}^{n} \dim H^i(\mathfrak{g}, K)$$

be the total cohomology of $\mathfrak{g}$. Show that for $n = 4m$ or $n = 4m + 2$ we have $\sigma(\mathfrak{g}) \equiv 0 \bmod 4$.

Exercise 33. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension $n \geq 2$ over a field K of characteristic not 2. Show that for $n = 4m + 1$ we also have $\sigma(\mathfrak{g}) \equiv 0 \bmod 4$.

Exercise 34. Give examples of n-dimensional nilpotent Lie algebras over a field K of characteristic zero with $n \equiv 3 \bmod 4$ such that $\sigma(\mathfrak{g}) \not\equiv 0 \bmod 4$.

Exercise 35. Let $\mathfrak{h}_m(K)$ be the $(2m + 1)$-dimensional Heisenberg Lie algebra over a field K. Assume that K has characteristic zero. Show that the Betti numbers of $\mathfrak{h}_m$ are not unimodal for $m \geq 4$.

Exercise 36. Find a nilpotent Lie algebra $\mathfrak{g}$ over a field of characteristic 2, such that the Betti numbers of $\mathfrak{g}$ are not log-concave.

Exercise 37. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero such that the Betti numbers of positive degree vanish, i.e., $H_i(\mathfrak{g}) = 0$ for all $i \geq 1$. Show that $\mathfrak{g} = 0$.

Due: January 30, 2023