Streamers in Gas Discharge Devices

Anne Nouri[1] and Christian Schmeiser[2]

Abstract. Streamers are ionization waves in gas discharge devices, modelled here as travelling wave solutions of a drift-diffusion model for the transport of electrons and ions. The existence of a certain type of small amplitude streamers is proved by applying results from the theory of singularly perturbed ordinary differential equations. Global information is obtained from a large activation energy analysis.

Zusammenfassung. Streamer sind Ionisierungswellen in Gasentladungsröhren, die hier durch wandernde Wellenlösungen eines Drift-Diffusionsmodelles für den Transport von Elektronen und Ionen beschrieben werden. Die Existenz von Wellen eines bestimmten Typs mit kleiner Amplitude wird mit Hilfe von Resultaten über singulär gestörte gewöhnliche Differentialgleichungen bewiesen. Globale Information liefert eine Störungsanalyse für große Aktivierungsenergie.

Key words: Gas discharge, travelling waves, singular perturbations

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[1]Laboratoire de Mathématiques, Université de Nice, France
[2]Institut für Angewandte und Numerische Mathematik, TU Wien, Austria.

1 Introduction

The streamer mechanism of electric discharge in gases is the subject of many investigations [1], [2], [5], [7], [8]. A streamer is a thin highly conducting plasma filament that grows at high speed through ionization in the strong electric field near its head. Although the physical processes connected with streamer propagation are well known, there does not exist a satisfactory qualitative theory of this phenomenon. Numerical simulations of streamer evolution have been performed [1], [8]. Taking them into account, the purpose of this paper is to use a simplified model in order to establish rigorously the existence of small amplitude streamers, described mathematically by travelling waves. After describing the model in Section 2, Section 3 is devoted to an existence proof for small amplitude streamers connecting specific stationary states. The streamer solutions are constructed by formal asymptotic methods. This approach is justified by an application of results from a geometric theory for singularly perturbed ordinary differential equations developped by Fenichel [4] and Szmolyan [11]. A drawback of these results is that they do not cover the commonly used models for the ionization rate. However, the assumptions permit a detailed analysis.

For more realistic models a formal asymptotic anaylsis is carried out in Section 4 assuming that a dimensionless parameter measuring the strength of the ionization rate is large. The considered ionization rate has the form of an Arrhenius term in chemical kinetics and the analysis is formally similar to a well established procedure called large activation energy asymptotics. It reveals some global information such as details about the structure of the ionization wave as well as the result that streamers exist only up to a certain strength. The results of this last section are illustrated by numerical computations.

2 The Model

A physically sound way of representing the ion and electron kinetics in a weakly ionized gas consists in using the Boltzmann transport equation. It is an integro-differential equation in a 7-dimensional space (the time and the 6-dimensional position-velocity phase-space). Its solutions yield the particle velocity distribution function. The excitation, dissociation and ionization rates can be deduced from it. The charged particle motion is coupled with the electric field, and the transport equations for each species of particles must be solved coupled to the Poisson equation for the electric field. Hence such a kinetic model is difficult to handle, theoretically as well as numerically. Therefore fluid models, where the particle transport is characterized by mean values such as densities, drift velocities, ..., are also studied by physicists [5], [8]. One of the simplest uses the drift-diffusion model for the flow of electrons as well as ions. The continuity equation for the electrons is

$$n_t - \nabla \cdot (D_n \nabla n - \mu_n n \nabla \varphi) = \nu n,$$

where n is the electron density, D_n the diffusion coefficient for electrons, μ_n the electron mobility, φ the electric potential, and ν the ionization rate. In order to close the system, it is assumed that ν is a function of the modulus $|E|$, of the electric field. ν is a smooth function, typically given by

$$\nu(w) = A e^{-(E_c/w)^\beta}, \qquad A, E_c, \beta > 0. \tag{2.1}$$

An analogous model is used for the tranport of the ions. Then the electron density n, the ion density p and the electric potential φ are solutions of the equations

$$\begin{aligned} n_t - \nabla \cdot (D_n \nabla n - \mu_n n \nabla \varphi) &= \nu(|\nabla \varphi|) n, & (2.2) \\ p_t - \nabla \cdot (D_p \nabla p + \mu_p p \nabla \varphi) &= \nu(|\nabla \varphi|) n, & (2.3) \end{aligned}$$

as well as the Poisson equation

$$\Delta \varphi = -\frac{q}{\varepsilon_0}(p - n). \tag{2.4}$$

Here, q is the elementary charge, ε_0 is the permittivity of the medium, μ_p is the ion mobility, and D_p the diffusion coefficient of the ions. Einstein's relations state that

$$D_n = U_T \mu_n, \qquad D_p = U_T \mu_p,$$

where the thermal voltage U_T is given by

$$U_T = \frac{k_B T}{q},$$

with Boltzmann's constant k_B and the device temperature T. An existence result for initial-boundary value problems for systems almost identical to (2.2)–(2.4) arising in semiconductor device simulation can be found in [10]. Only the source term in (2.2), (2.3) is different, but this does not require substantial modifications of the proof.

Before we start our analysis we introduce the electric field $E = -\nabla\varphi$ and a scaling of (2.2)–(2.4). In particular, we introduce dimensionless variables by

$$E \to E_c E, \quad n \to \frac{\varepsilon_0 E_c^2}{U_T q} n, \quad p \to \frac{\varepsilon_0 E_c^2}{U_T q} p, \quad x \to \frac{U_T}{E_c} x, \quad t \to \frac{U_T}{\mu_n E_c^2} t,$$
$$\nu \to \frac{\mu_n E_c^2}{U_T} \nu, \quad \mu_p \to \mu_n \mu_p.$$

The scaled version of (2.2)–(2.4) then reads

$$\begin{aligned} n_t - \nabla \cdot (\nabla n + nE) &= \nu(|E|)n\,, \\ p_t - \mu_p \nabla \cdot (\nabla p - pE) &= \nu(|E|)n\,, \\ \nabla \cdot E &= p - n\,. \end{aligned} \tag{2.5}$$

In this paper we are looking for streamers, which — to our knowledge — have not been analyzed mathematically so far. Numerical computations [2] in an axisymmetric geometry show that the densities and the electric field vary very little radially, in a neighbourhood of the axis of propagation. Hence, as a first step in the study of streamers, we restrict our attention to such a neigbourhood, and look for all unknowns as plane waves with constant velocity, i.e functions of the variable $\xi = z - st$, where z denotes the longitudinal space variable and s the speed of the wave. With derivation with respect to ξ denoted by $'$, we therefore consider the following system of ordinary differential equations:

$$\begin{aligned} -sn' - (n' + nE)' &= \nu(|E|)n\,, \\ -sp' - \mu_p(p' - pE)' &= \nu(|E|)n\,, \\ E' &= p - n\,. \end{aligned} \tag{2.6}$$

In particular, we are interested in heteroclinic orbits, i.e. solutions connecting stationary points of the system (2.6). The following result is obvious:

Lemma 2.1 *Assume $\nu(0) = 0$ and $\nu(w) > 0$ for $w \neq 0$. Then a stationary point (n, p, E) of (2.6) either satisfies $n = p = 0$ or $n = p$, $E = 0$.*

In the following, streamer solutions will be constructed with vanishing field at $\xi = -\infty$ and vanishing electron and ion densities at $\xi = \infty$. The opposite case does not have to be considered separately, since the problem has a reflection symmetry: The equations (2.6) are invariant under the transformation

$$\xi \to -\xi, \qquad E \to -E, \qquad s \to -s.$$

3 Small Amplitude Streamers

The aim of this section is to establish the existence of small amplitude streamers, i.e., solutions of (2.6) satisfying

$$\begin{array}{ll} n(-\infty) = p(-\infty) = \varepsilon n_- > 0\,, & E(-\infty) = 0\,, \\ n(\infty) = p(\infty) = 0\,, & E(\infty) = \varepsilon E_+\,, \end{array} \tag{3.1}$$

with $0 < \varepsilon \ll 1$. Introducing the electron current density $J_n = n' + nE$, (3.1) implies

$$J_n(-\infty) = J_n(\infty) = 0\,. \tag{3.2}$$

Also the order of the problem can be reduced by the observation that the difference of the electron and ion continuity equations in (2.6) can be integrated:

$$s(p-n) - J_n + \mu_p(p' - pE) = 0\,,$$

where the constant of integration has been determined from (3.1), (3.2). Being interested in small amplitude streamers, we introduce the rescaling

$$n \to \varepsilon n, \quad p \to \varepsilon p, \quad J_n \to \varepsilon^2 J_n, \quad E \to \varepsilon E, \quad \xi \to \frac{\xi}{\varepsilon}.$$

Now the traveling wave equations can be written as the first order system

$$\begin{array}{rcl} n' &=& J_n - nE\,, \\ \varepsilon p' &=& \varepsilon pE + \frac{s}{\mu_p}(n-p) + \frac{\varepsilon}{\mu_p} J_n\,, \\ \varepsilon J_n' &=& s(nE - J_n) - \frac{\nu(\varepsilon|E|)}{\varepsilon} n\,, \\ \varepsilon E' &=& p - n\,, \end{array} \tag{3.3}$$

subject to

$$\begin{array}{lll} n(-\infty) = p(-\infty) = n_- \,, & J_n(-\infty) = 0 \,, & E(-\infty) = 0 \,, \\ n(\infty) = p(\infty) = 0 \,, & J_n(\infty) = 0 \,, & E(\infty) = E_+ \,. \end{array} \tag{3.4}$$

We assume the ionization rate to be of the form

$$\nu(w) = \nu_1 w + O(w^2), \quad \text{as } w \to 0, \qquad \nu_1 > 0 \,. \tag{3.5}$$

Note that this excludes the model (2.1), where the function ν is flat to arbitrary order at $w = 0$.

The system (3.3) is singularly perturbed since three of the four equations degenerate to algebraic equations in the formal limit $\varepsilon \to 0$. There is one slow variable (n) and three fast variables (p, J_n and E). However, the system is not in standard form, because two of the limiting equations are identical. Thus, the limiting solution cannot be completely determined from these equations. In the language of singular perturbation theory, (3.3) is called *singular* singularly perturbed [9]. A transformation to standard form is achieved by replacing p by the new variable $z = \mu_p p + sE$. The transformed system reads

$$\begin{array}{rcl} n' &=& J_n - nE \,, \\ z' &=& (z - sE)E + J_n \,, \\ \varepsilon J_n' &=& s(nE - J_n) - \frac{\nu(\varepsilon|E|)}{\varepsilon} n \,, \\ \varepsilon E' &=& \frac{z - sE}{\mu_p} - n \,, \end{array} \tag{3.6}$$

with two slow and two fast variables. For the new variable z we have

$$z(-\infty) = \mu_p n_- \,, \qquad z(\infty) = sE_+ \,.$$

We now consider the reduced system, i.e. the formal limit of (3.6) as $\varepsilon \to 0$. After elimination of z and J_n and some straightforward manipulations we obtain

$$\begin{array}{rcl} n' &=& -\frac{\nu_1}{s} n|E| \,, \\ E' &=& \frac{nE}{s}\left(1 + \mu_p - \frac{\nu_1(1-\mu_p)}{s}\mathrm{sign}E\right) \,, \end{array} \tag{3.7}$$

with

$$z = \mu_p n + sE \,, \qquad J_n = nE\left(1 - \frac{\nu_1}{s}\mathrm{sign}E\right) \,. \tag{3.8}$$

The reduced system (3.7) possesses two curves of equilibria given by $(n, E) = (n_-, 0)$ and $(n, E) = (0, E_+)$. These correspond to the curves of equilibria $(n, z, J_n, E) = (n_-, \mu_p n_-, 0, 0)$ and $(n, z, J_n, E) = (0, sE_+, 0, E_+)$ of the full problem (3.6).

It is obvious from (3.7) that n and E are monotone and, in particular, that E does not have any sign changes. Therefore $\mathrm{sign} E(\xi) = \mathrm{sign} E_+$ holds. Because of $n(-\infty) > n(\infty)$ the electron density is monotonically decreasing, implying $s > 0$ by (3.7). The fact that the streamer moves to the right is physically obvious, since a left moving wave would correspond to recombination of electrons and ions whereas our model does not incorporate recombination effects.

The system (3.7) implies

$$\frac{dE}{dn} = \frac{1-\mu_p}{s} - \frac{1+\mu_p}{\nu_1} \mathrm{sign} E_+ \,,$$

and solutions lie on straight lines in the n-E-plane. Considering the conditions at $\xi = \infty$ we obtain

$$\frac{E - E_+}{n} = \frac{1-\mu_p}{s} - \frac{1+\mu_p}{\nu_1} \mathrm{sign} E_+ \,.$$

Evaluation at $\xi = -\infty$ gives an equation relating n_-, E_+ and the wave velocity s. We compute

$$s = \frac{n_- \nu_1 (1-\mu_p)}{(1+\mu_p) n_- \mathrm{sign} E_+ - \nu_1 E_+} \,. \tag{3.9}$$

In the following we use the physically reasonable assumption $\mu_p < 1$ meaning that the mobility of the ions is smaller than that of the electrons. Since s has to be positive we obtain the requirement

$$(1+\mu_p) n_- \mathrm{sign} E_+ > \nu_1 E_+ \tag{3.10}$$

for the data n_- and E_+. Streamer solutions of the reduced problem exist if the data lie in one of the two sectors of the n_--E_+-half plane determined by $(1+\mu_p)n_- > \nu_1 E_+ > 0$ and $-(1+\mu_p)n_- > \nu_1 E_+$ (see Fig. 1). For fixed electron (and ion) density behind the streamer the following statement holds: If the electric field ahead of the streamer is positive it cannot be too large, if it is negative it cannot be too small.

Figure 1: Regions of the n_--E_+-half plane where streamers exist.

The system (3.6) fits into the framework of Fenichel's geometric theory for singularly perturbed ODEs [4] (see also Szmolyan's work [11]). An application of this theory justifies our formal asymptotics:

Theorem 3.1 *Assume that $\mu_p < 1$ holds, that the ionization rate satisfies (3.5), and that (3.10) holds. Then, for ε small enough, there exists a velocity s such that (3.3), (3.4) has a solution. The limit of the velocity as $\varepsilon \to 0$ is given by (3.9).*

Proof. We consider the system (3.6) for given positive s. The Jacobian of the right hand side of the third and fourth equations in (3.6) with respect to the fast variables J_n and E is given by

$$\begin{pmatrix} -s & n(s - \nu'(\varepsilon|E|)\mathrm{sign}E) \\ 0 & -s/\mu_p \end{pmatrix},$$

and has the eigenvalues $-s < 0$ and $-s/\mu_p < 0$. This is sufficient for the existence of a two dimensional invariant manifold $\mathcal{S}_\varepsilon$ of (3.6) (and, therefore, of (3.3)) for small ε which is close to the reduced manifold $\mathcal{S}_0$ determined by (3.8). All structurally stable properties of the flow on $\mathcal{S}_0$ determined by the reduced equations (3.7) persist in the flow on $\mathcal{S}_\varepsilon$. This is a consequence of theorem 2.1 in [11].

Two difficulties arise when the general results in [11] are applied two our problem. First, the lack of smoothness of the right hand side of (3.6) caused by the term $|E|$. This problem can be circumvented by first applying the theory to the two systems where $|E|$ is replaced by E and, respectively, $-E$ and then using the results only for positive and, respectively, negative E. This approach works since the heteroclinic orbits of the reduced problem are transversal to the $(E=0)$-hyperplane, a property which is inherited by the flow on $\mathcal{S}_\varepsilon$.

The second difficulty is that the existence of curves of equilibria is not a structurally stable property of the flow on $\mathcal{S}_0$. However, the persistence of this structure in the perturbed problem (3.6) is obvious. This observation together with the above mentioned theorem implies that the flow on $\mathcal{S}_\varepsilon$ is determined by a system of the form

$$\begin{aligned} n' &= -\tfrac{\nu_1}{s} n|E|(1+f(n,E,\varepsilon)), \\ E' &= \tfrac{nE}{s}\left(1+\mu_p-\tfrac{\nu_1(1-\mu_p)}{s}\mathrm{sign}E\right)(1+g(n,E,\varepsilon)), \end{aligned} \tag{3.11}$$

with smooth functions f and g, satisfying $f(n,E,0)=g(n,E,0)=0$. From (3.11) it is clear that all the heteroclinic orbits of the reduced problem constructed above are limits of heteroclinic orbits of the full problem as $\varepsilon \to 0$ implying the statements of the theorem. □

4 Large Activation Energy Asymptotics

Now we consider the more realistic model (2.1) with $\beta = 1$ for the ionization rate. The scaled version of ν then reads

$$\nu(w) = \gamma e^{-1/w} \tag{4.1}$$

with the dimensionless parameter

$$\gamma = \frac{AU_T}{\mu_n E_c^2}.$$

In this section we take into account the fact that ions are much heavier than electrons by neglecting the transport of ions. After an integration similar to the preceeding section, the travelling wave equations (2.6) with (4.1) and $\mu_p = 0$ can be written as the first order system

$$\begin{aligned} n' &= J_n - nE, \\ J_n' &= s(nE - J_n) - \gamma e^{-1/|E|} n, \\ E' &= J_n/s, \end{aligned} \tag{4.2}$$

where $J_n = n' + nE$ is the flux of electrons relative to an observer moving with speed s, as above.

In the following we consider the case where γ is large and perform a formal asymptotic analysis. This is motivated by a similarity of our problem with mathematical models for combustion waves [6]. In particular, the model (4.1) for the ionization rate has the form of an Arrhenius term in chemical kinetics with the electric field strength playing the role of temperature. In this context, the critical field strength E_c in (2.1) corresponds to the activation energy.

A small parameter $\delta > 0$ is introduced by

$$\gamma = \delta^2 e^{1/\delta},$$

where the smaller solution $\delta \sim (\ln \gamma)^{-1}$ is taken. After the rescaling

$$n \to \delta^3 n, \quad J \to \delta^4 J, \quad E \to \delta E, \quad s \to \delta s, \quad \xi \to \delta^{-2}\xi,$$

(4.2) reads

$$\begin{aligned} \delta n' &= J_n - nE\,, \\ \delta J_n' &= s(nE - J_n) - \exp\left(\tfrac{1}{\delta}\left(1 - \tfrac{1}{|E|}\right)\right) n\,, \\ E' &= J_n/s\,. \end{aligned} \tag{4.3}$$

Comparison of the ionization rate in (4.3) with the unscaled version (2.1) shows that the scaled activation energy is now given by $1/\delta$. Therefore an asymptotic analysis for $\delta \to 0$ is — in the context of combustion waves — called 'large activation energy asymptotics' [6].

As the systems (3.3) and (3.6) in the preceeding section, (4.3) is also singularly perturbed. However, (4.3) cannot be written in a standard form because of the strongly nonlinear behaviour of the ionization rate. It takes exponentially large values for $|E| > 1$ and exponentially small values for $|E| < 1$. Therefore, we restrict ourselves to a formal asymptotic analysis.

As in the preceeding section we construct solutions with an ionized, field-free state left of the wave:

$$n(-\infty) = n_- > 0\,, \qquad J_n(-\infty) = E(-\infty) = 0\,.$$

We consider (4.3) with given wave velocity s, subject to the above boundary conditions at $\xi = -\infty$. In this problem we perform the formal limit $\delta \to 0$. The limiting solution will be denoted by $(\overline{n}, \overline{J}_n, \overline{E})$.

Because of the boundary conditions we expect a region with $|\overline{E}| < 1$ in a neighbourhood of $\xi = -\infty$. There,

$$\overline{J}_n = \overline{n}\overline{E}\,, \qquad \overline{J}_n' + s\overline{n}' = 0$$

holds, where the second equation is obtained from a linear combination of the first and second equations in (4.3). Integration gives

$$\overline{J}_n + s(\overline{n} - n_-) = 0\,.$$

After elimination of $\overline{n}$ and $\overline{J}_n$, the third equation in (4.3) becomes

$$\overline{E}' = \frac{n_-\overline{E}}{s + \overline{E}}\,. \tag{4.4}$$

Only for $s > 0$ this equation has nontrivial solutions vanishing at $\xi = -\infty$. It was also a result of the preceeding section that the wave travels to the right. Equation (4.4) remains valid as long as $|\overline{E}| < 1$ holds. We construct solutions with $|\overline{E}| = 1$ at $\xi = 0$, i.e.,

$$\overline{E}(0) = \sigma \qquad \text{with } \sigma = 1 \text{ or } \sigma = -1\,.$$

Such solutions exist iff

$$s > \max\{0, -\sigma\} \tag{4.5}$$

holds which we assume in the following. Note that $-\sigma$ is the drift velocity of the electrons at $\xi = 0$. The above condition, thus, means that the wave velocity has to be larger than the drift velocity.

The solution of (4.4) is given implicitly by

$$s\ln|\overline{E}| + \overline{E} - \sigma = n_-\xi\,, \qquad \text{for } \xi < 0\,. \tag{4.6}$$

We propose that limiting solutions have jumps in the n- and J_n-components and a corner in the E-component at $\xi = 0$:

$$\overline{n} = \begin{cases} \frac{n_- s}{s+\overline{E}}\,, & \xi < 0, \\ 0\,, & \xi > 0, \end{cases} \qquad \overline{J}_n = \begin{cases} \frac{n_- s\overline{E}}{s+\overline{E}}\,, & \xi < 0, \\ 0\,, & \xi > 0, \end{cases}$$

$$\overline{E} = \begin{cases} \text{given by (4.6)}, & \xi < 0, \\ \sigma\,, & \xi > 0. \end{cases}$$

A (formal) justification of this proposal can be provided by the construction of layer solutions smearing out the jumps. A fast independent variable is introduced by

$$\eta = \frac{\xi}{\delta}\,.$$

Also, in the layer the values of the field are expected to be close to σ: $E = \sigma(1+\delta\mathcal{E})$. The formal limits of the solution components in the layer are denoted by

$$\hat{n}(\eta) = \lim_{\delta\to 0} n(\delta\eta;\delta)\,, \qquad \hat{J}_n(\eta) = \lim_{\delta\to 0} J_n(\delta\eta;\delta)\,, \qquad \hat{\mathcal{E}}(\eta) = \lim_{\delta\to 0} \mathcal{E}(\delta\eta;\delta)\,.$$

The layer equations are

$$\begin{aligned}
\frac{d\hat{n}}{d\eta} &= \hat{J}_n - \hat{n}\sigma\,,\\
\frac{d\hat{J}_n}{d\eta} &= s(\hat{n}\sigma - \hat{J}_n) - e^{\hat{\mathcal{E}}}\hat{n}\,,\\
\frac{d\hat{\mathcal{E}}}{d\eta} &= \frac{1}{s\sigma}\hat{J}_n\,,
\end{aligned}$$

subject to the auxiliary conditions

$$\begin{aligned}
\hat{n}(-\infty) &= \tfrac{n_- s}{s+\sigma}\,, & \hat{n}(\infty) &= 0\,,\\
\hat{J}_n(-\infty) &= \tfrac{n_- s\sigma}{s+\sigma}\,, & \hat{J}_n(\infty) &= 0\,,\\
\hat{\mathcal{E}}(-\infty) &= -\infty\,, & \hat{\mathcal{E}}(\infty) &= \mathcal{E}_+\,,
\end{aligned}$$

where the value of the field ahead of the streamer is $E_+ = \sigma(1+\delta\mathcal{E}_+)$. The layer problem is brought into a more managable form by the transformation

$$\hat{n} = \frac{n_- s}{s+\sigma}u\,, \qquad \hat{J}_n = \frac{n_- s\sigma}{s+\sigma}v\,, \qquad \hat{\mathcal{E}} = \ln w\,.$$

Now the differential equations read

$$\begin{aligned}
\frac{du}{d\eta} &= \sigma(v-u)\,,\\
\frac{dv}{d\eta} &= s(u-v) - \sigma u w\,,\\
\frac{dw}{d\eta} &= \alpha v w\,, \qquad \text{with } \alpha = \frac{n_-}{s+\sigma} > 0\,,
\end{aligned} \tag{4.7}$$

subject to the boundary conditions

$$\begin{array}{rclrcl} u(-\infty) & = & 1\,, & u(\infty) & = & 0\,, \\ v(-\infty) & = & 1\,, & v(\infty) & = & 0\,, \\ w(-\infty) & = & 0\,, & w(\infty) & = & \beta = \exp(\mathcal{E}_+)\,. \end{array} \tag{4.8}$$

Thus, we are looking for heteroclinic orbits of (4.7) connecting the curves of equilibria $\{(u,u,0)\}$ and $\{(0,0,w)\}$. The linearized problem at $(1,1,0)$ has the eigenvalues

$$\lambda_1 = -s - \sigma < 0\,, \qquad \lambda_2 = 0\,, \qquad \lambda_3 = \alpha > 0\,.$$

An eigenvector corresponding to λ_3 is given by $r_3 = (-1, -\sigma(\sigma+\alpha), \alpha(\alpha+s+\sigma))$. Note that the w-component of r_3 is positive. The differential equation for w shows that this solution component does not change sign and, thus, that $w > 0$ holds along the heteroclinic orbit. This implies that the heteroclinic orbit (if it exists) must follow the one-dimensional unstable manifold of the point $(1,1,0)$ in the direction of r_3.

The eigenvalues of the linearization at $(0,0,\beta)$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left(-s - \sigma \pm \sqrt{(s+\sigma)^2 - 4\beta}\right)\,, \qquad \lambda_3 = 0\,.$$

Since $\mathrm{Re}(\lambda_{1,2}) < 0$ holds, the curve $\{(0,0,w)\}$ is locally attracting. However, for

$$\beta > \frac{(s+\sigma)^2}{4}$$

the eigenvalues $\lambda_{1,2}$ are complex, meaning that u and v oscillate as they tend to zero. Since u is a rescaled version of the electron density, such a solution (taking negative values) is physically meaningless.

The linearization above and the results of the preceeding section suggest that in the problem (4.7), (4.8) one of the three parameters s, α and β has to be considered as unknown. It seems convenient to prescribe the values of s and α. Then we can follow the unstable manifold of the point $(1,1,0)$ in the direction of r_3 and ask the question if this trajectory is attracted by the curve $\{0,0,w\}$. Lacking an analytical answer to this question we are having recourse to numerical experiments using the software package 'PhasePlane' ([3]) for the simulation of dynamical systems.

Figure 2: a) Electron density and b) electric field of streamer solutions with negative field for different ionization levels.

In these experiments we have chosen fixed values for the wave velocity s (satisfying (4.5)) and varied α. For α small enough the solution starting at $(1, 1, 0)$ converged to a point $(0, 0, \beta(\alpha))$. The computed results indicate that $\beta(0) = 0$ holds and that the function $\beta(\alpha)$ is increasing for small arguments. It also turns out that for large enough values of α, β eventually exceeds the critical value $(s + \sigma)^2/4$. In these cases, the solution component u takes negative values.

Numerical computations have also been carried out with the full system (4.3). As an illustration Fig. 2 shows plots of the electron density and of the electric field for different values of n_- and for $E_+ < 0$. The layer structure is clearly visible. Also the oscillating behaviour to the right of the wave for larger values of n_- can be seen. Since n takes negative values in these cases, these solutions do not have any physical significance. In Fig. 3 the results of a similar series of computations is shown, however with $E_+ > 0$ and with a smaller value of the perturbation parameter δ. In agreement with our asymptotics the density to the left of the layer is now decreasing. The values of the field to the right of the wave are now closer to each other (and

Figure 3: a) Electron density and b) electric field of streamer solutions with positive field for different ionization levels.

closer to the value 1), as can be expected for a smaller value of δ from the asymptotics.

As a conclusion we emphasize that the results of this and the preceeding sections agree in so far as for given wave velocity s the value n_- of the electron (and ion) density to the left of the wave and the value E_+ of the field to the right of the wave lie on a curve through the origin. In the local analysis of section 3 this curve degenerates to a straight line. The results of section 4 show that for only a finite segment of this curve physically relevant solutions can be expected to exist.

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