Numerical Passage from Radiative Heat Transfer to Nonlinear Diffusion Models *

A. Klar[†] C. Schmeiser[‡]

Abstract

Radiative heat transfer equations including heat conduction are considered in the small mean free path limit. Rigorous results on the asymptotic procedure leading to the equilibrium diffusion equation for the temperature are given. Moreover, the nonlinear Milne problem describing the boundary layer is investigated and an existence result is proven. An asymptotic preserving scheme for the radiative transfer equations with the diffusion scaling is developed. The scheme is based on the asymptotic analysis. It works uniformly for all ranges of mean free paths. Numerical results for different physical situations are presented.

Keywords. radiative heat transfer, asymptotic analysis, nonlinear diffusion limit, Milne problem, asymptotic preserving methods, numerical methods for stiff equations

AMS subject classifications. 82C70, 65M06, 45K05

1 Introduction

Radiative transfer equations are classical in astrophysics and also used to describe a variety of other physical phenomena. For example, simulation of heat transfer in semitransparent materials such as glass is usually done on the basis of radiative transfer equations including a model for heat conduction. We refer to [17], [19], [23] for a detailed description of mathematical models and further references.

Macroscopic limits of radiative transfer models are obtained by an asymptotic analysis using the diffusion scaling and letting the Knudsen number (i.e.,

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†Fachbereich Mathematik, Technische Universität Darmstadt, 64289 Darmstadt, Germany, (`klar@mathematik.tu-darmstadt.de`).

‡Institut für Angewandte und Numerische Mathematik Technische Universität Wien, 1040 Wien, Austria, (`Christian.Schmeiser@tuwien.ac.at`).

the scaled mean free path) tend to zero. Typically, the limit equation is a nonlinear diffusion equation for the temperature; we refer to [13], [18], [12]. Rigorous results on the existence of solutions and on the macroscopic limit can be found, for example, in [2, 1].

If standard numerical methods are used to solve kinetic equations for small Knudsen numbers, usually, a very fine and expensive discretization depending on the mean free path is required due to the stiffness of the equations. This makes these schemes extremely time consuming. Therefore, numerical schemes have to be developed working uniformly for different regimes. In particular, it should be possible to chose the discretization size independent of the mean free path. In recent years there has been a lot of work on numerical methods for kinetic equations in stiff regimes. For example, stationary transport equations in the diffusion limit have been considered in [7, 15, 14, 16]. Nonstationary kinetic equations are treated in [4, 5, 9, 8, 21, 20, 10, 11].

In the present paper analytical and numerical investigations of the radiative heat transfer equations including a heat conduction term are presented. The macroscopic limit is rigorously justified and existence of a solution of the nonlinear Milne problem describing boundary layers is proven. A numerical method for the radiative transfer equations working uniformly for small Knudsen numbers is presented. The method is based on a semiimplicit discretization treating different time scales involved in the problem in a suitable way. As a result, spatial and temporal grids can be chosen independently of small Knudsen numbers. The main difficulties, compared to previous studies, originate from the nonlinearity of the collision operator and the inclusion of heat conduction as well as reflection-transmission boundary conditions. The numerical methods of [8, 11, 10] would require the solution of nonlinear problems in each time step. Here, a linearization procedure is used with the consequence that the main part of the implicit time stepping has the cost of solving a linear diffusion equation implicitly. Section 2 contains a description of the equations and the formal asymptotic procedure. The equations for inital and boundary layers are stated. Section 3 contains the rigorous results. Existence and uniqueness of the radiative heat transfer equations and convergence towards the diffusion equation are proven. Moreover, existence of a solution of the Milne problem is shown. Here the heat conduction term induces major difficulties since its inclusion inhibits the elimination of the nonlinearity. In Section 4 the numerical procedure is introduced. Time, space, and velocity discretizations motivated by the asymptotic analysis are presented in detail. Some numerical results are described in 5.

2 Radiative Heat Transfer Equations and Asymptotic Limits

In a domain $D \in \mathbb{R}^3$ we consider the radiative transfer equations including conductive heat transfer but without photon scattering. The space variable is denoted by $x \in D$, the direction by the unit vector $\Omega \in S^2$, S^2 the unit sphere of $\mathbb{R}^3$ and $t \in \mathbb{R}^+$ the time. For simplicity, only a one band approximation is considered. However, most of the results, in particular, the numerical algorithm presented in Section 4, can be extended to the general case.

We denote by $I = I(x, \Omega, t)$ the radiative intensity at x in the direction Ω and by $T(x,t)$ the temperature. Introducing the Knudsen number ε given by the relation of the inverse absorption cross-section to a characteristic length scale, the equations in nondimensional form are

$$\varepsilon^2 M \partial_t I_\varepsilon + \varepsilon \Omega \cdot \nabla_x I_\varepsilon = B(T_\varepsilon) - I_\varepsilon \,, \tag{1}$$

$$\varepsilon^2 \partial_t T_\varepsilon = \varepsilon^2 \Delta_x T_\varepsilon - (B(T_\varepsilon) - \langle I_\varepsilon \rangle) \,, \tag{2}$$

with the total intensity

$$\langle I \rangle (x,t) = \frac{1}{|S^2|} \int_{S^2} I(x, \Omega, t) d\Omega \,,$$

and the black body intensity

$$B(T) = \sigma T^4 \,, \tag{3}$$

where $M, \sigma > 0$ are dimensionless constants. We have to prescribe initial values

$$I_\varepsilon(x, \Omega, 0) = I_I(x, \Omega) \,, \qquad T_\varepsilon(x, 0) = T_I(x) \,. \tag{4}$$

Moreover, boundary conditions have to be prescribed: we consider reflection transmission boundary conditions. At $\hat{x} \in \partial D$ with outward unit normal n, we consider

$$I_\varepsilon(\hat{x}, \Omega, t) = \rho(n \cdot \Omega) I_\varepsilon(\hat{x}, \Omega', t) + [1 - \rho(n \cdot \Omega)] I_B(\hat{x}, \Omega), \quad n \cdot \Omega < 0, \tag{5}$$

where

$$\Omega' = \Omega - 2n(n \cdot \Omega)$$

is the reflection of Ω in the tangent plane to ∂D. The reflectivity ρ, $0 \leq \rho \leq 1$, depends on the incidence angle and I_B denotes the radiative intensity transmitted into the medium from the outside. Moreover, we impose a boundary condition on the temperature

$$T_\varepsilon(\hat{x}, t) = T_B(\hat{x}). \tag{6}$$

For ε small the approximation of (1,2) that is usually used is the diffusion approximation. It is obtained by the following asymptotic procedure: write

$$\begin{aligned} I_\varepsilon &= I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + \cdots \\ T_\varepsilon &= T_0 + \varepsilon^2 T_2 + \cdots \end{aligned}$$

Substituting this ansatz into (1, 2) one obtains collecting the terms of the same order in ε:

$$\begin{aligned} O(1): &\quad I_0 = \langle I_0 \rangle = B(T_0) \\ O(\varepsilon): &\quad \Omega \cdot \nabla_x I_0 = -I_1 \,, \ \langle I_1 \rangle = 0 \end{aligned}$$

A combination of (1) and (2) gives the energy conservation equation

$$\partial_t(M\langle I_\varepsilon\rangle + T_\varepsilon) + \nabla_x \cdot \left(\frac{\langle \Omega I_\varepsilon\rangle}{\varepsilon} - \nabla_x T_\varepsilon\right) = 0\,, \tag{7}$$

with the limit

$$\partial_t(M\langle I_0\rangle + T_0) + \nabla_x \cdot (\langle \Omega I_1\rangle - \nabla_x T_0) = 0\,.$$

This yields

$$\partial_t(MB(T_0) + T_0) = \nabla_x \cdot (\nabla_x T_0 + D\nabla_x B(T_0))\,, \tag{8}$$

with $D = \langle \Omega \otimes \Omega\rangle = \frac{1}{3}Id$ (where Id denotes the 3 by 3 identity matrix), or

$$(MB'(T_0) + 1)\partial_t T_0 = \nabla_x \cdot [(B'(T_0)/3 + 1)\nabla_x T_0]\,. \tag{9}$$

The above is only valid away from initial- and boundary layers. Initial conditions are found by solving an initial layer problem:

$$\begin{aligned} M\partial_\tau \hat I &= B(\hat T) - \hat I\,, \\ \partial_\tau \hat T &= \langle \hat I\rangle - B(\hat T)\,, \\ \hat I(x,\Omega,0) &= I_I(x,\Omega)\,, \qquad \hat T(x,0) = T_I(x)\,, \end{aligned}$$

where the layer solutions are functions of the fast time variable $\tau = t/\varepsilon^2$. Existence and uniqueness of the solution of the layer problem are easily shown as well as convergence for $\tau \to \infty$ with

$$\begin{aligned} &\hat I(x,\Omega,\infty) = B(\hat T(x,\infty))\,, \\ &MB(\hat T(x,\infty)) + \hat T(x,\infty) = M\langle I_I\rangle(x) + T_I(x)\,. \end{aligned}$$

Because of the monotonicity of B, $\hat T(x,\infty)$ can be determined uniquely from the second equation. The initial condition for the diffusion equation (8) is given by

$$T_0(x,0) = \hat T(x,\infty)\,.$$

Boundary conditions for the diffusion equation are found by a boundary layer analysis. For $\hat x \in \partial D$ a fast variable y is introduced by stretching in normal direction to the boundary: $\varepsilon y = (\hat x - x)\cdot n$. This leads to a one-dimensional nonlinear Milne problem at $\hat x$ for $T = T(y), I = I(y,\Omega)$. Defining $\mu = -\Omega \cdot n$ we have:

$$\begin{aligned} \mu\partial_y I &= B(T) - I\,, \\ \partial_y^2 T &= B(T) - \langle I\rangle\,, \end{aligned} \tag{10}$$

with $y \in [0,\infty)$. Boundary conditions have to be imposed at $y = 0$:

$$\begin{aligned} T(0) &= T_B(\hat x)\,, \\ I(0,\Omega) &= \rho(\mu)I(0,\Omega') + [1-\rho(\mu)]I_B(\hat x,\Omega)\,, \quad \mu > 0\,. \end{aligned} \tag{11}$$

One is interested in solutions that are bounded at infinity. Existence of solutions of the Milne problem is shown in the following section.

The boundary conditions for (8) are found by determining $T(\infty)$ and $I(\infty, \Omega) = B(T(\infty))$ for each $\hat{x}$. Then one uses $T_0(\hat{x}, t) = T(\infty)$. We mention that, in general, for $I_B \neq B(T_B)$, one has $T(\infty) \neq T_B(\hat{x})$.

The diffusion equation is a parabolic equation for the total energy density $T_0 + MB(T_0)$ with density dependent diffusivity. Existence and uniqueness for the initial-boundary value problem follows from standard results.

We refer to [13], [17], [19], [22], [23] and [12] for the physical background and a detailed description of the equations. Rigorous mathematical results for related problems can be found, for example, in [2, 1], where inflow boundary conditions are considered ($\rho(n \cdot \Omega) = 0$), and the heat conduction term in (2) does not appear.

3 Existence and Convergence Results

We prove an existence and uniqueness results for the problem (1), (2), (4), (5), (6) under the following assumptions:

A1: $D \subset \mathbb{R}^3$ is bounded and ∂D is smooth.

A2: $I_I \in L^\infty(D \times S^2)$, $T_I \in L^\infty(D)$.

A3: $I_B \in L^\infty(\partial D \times S^-)$ with $\partial D \times S^- = \{(\hat{x}, \Omega) : \ \hat{x} \in \partial D, \ n(\hat{x}) \cdot \Omega < 0\}$, $T_B \in H^{1/2}(\partial D) \cap L^\infty(\partial D)$.

A4: There are constants γ_1, γ_2 such that

$$0 \leq \gamma_1 \leq T_I, T_B \leq \gamma_2 \,, \qquad B(\gamma_1) \leq I_I, I_B \leq B(\gamma_2) \,.$$

Theorem 1
Let the assumptions A1–A4 hold. Then there exists a unique solution $(I_\varepsilon, T_\varepsilon) \in L^\infty(D \times S^2 \times \mathbb{R}^+) \times L^\infty(D \times \mathbb{R}^+)$ of problem (1), (2), (4), (5), (6) with

$$\gamma_1 \leq T_\varepsilon(x,t) \leq \gamma_2 \,, \qquad B(\gamma_1) \leq I_\varepsilon(x, \Omega, t) \leq B(\gamma_2) \,.$$

Proof: The solution will be constructed by a fixed point iteration. For $T \in \mathcal{S} := \{T \in L^\infty(D \times \mathbb{R}^+) : \ \gamma_1 \leq T \leq \gamma_2\}$, we define I as the solution of the linear transport problem

$$\begin{aligned}
&\varepsilon^2 M \partial_t I + \varepsilon \Omega \cdot \nabla_x I = B(T) - I \,, \\
&I = I_I \,, \qquad \text{for } t = 0 \,, \\
&I(\hat{x}, \Omega, t) = \rho(n(\hat{x} \cdot \Omega) I(\hat{x}, \Omega', t) + [1 - \rho(n(\hat{x} \cdot \Omega)] I_B(\hat{x}, \Omega) \,, \\
&\qquad\qquad \text{for } (\hat{x}, \Omega, t) \in \partial D \times S^- \times \mathbb{R}^+ \,.
\end{aligned} \tag{12}$$

Existence of a unique solution is a standard result. A straightforward maximum principle argument implies $B(\gamma_1) \le I \le B(\gamma_2)$. The definition of the fixed point operator is completed by solving the semilinear parabolic problem

$$\begin{aligned}
&\varepsilon^2 \partial_t \theta = \varepsilon^2 \Delta_x \theta - B(\theta) + \langle I \rangle \,, \\
&\theta = T_I \,, \qquad \text{for } t = 0 \,, \\
&\theta(\hat{x}, t) = T_B(\hat{x}) \,, \quad \text{for } (\hat{x}, t) \in \partial D \times \mathrm{I\!R}^+ \,,
\end{aligned} \tag{13}$$

and setting $F(T) = \theta$. Again, the unique solvability of the problem for θ is standard, and $\theta \in \mathcal{S}$ is a consequence of the maximum principle. Thus, $F : \mathcal{S} \to \mathcal{S}$ is well defined.

The proof will be completed by showing that for $\tau > 0$ small enough, F is a contraction in the space $L^\infty(0, \tau; L^2(D))$: let $T_1, T_2 \in \mathcal{S}$ and set $r = I_1 - I_2$, where I_i is the solution of (12) with T replaced by T_i. Then r solves (12) with $B(T)$ replaced by $B(T_1) - B(T_2)$ and with $I_I = I_B = 0$. Multiplication of the transport equation by r and integration over $D \times S^2$ gives

$$\begin{aligned}
&\frac{\varepsilon^2 M}{2} \frac{d}{dt} \int_D \langle r^2 \rangle dx + \frac{\varepsilon}{2} \int_{\partial D} n \cdot \langle \Omega r^2 \rangle d\sigma \\
&= \int_D [(B(T_1) - B(T_2)) \langle r \rangle - \langle r^2 \rangle] dx \,,
\end{aligned}$$

where $d\sigma$ is the two-dimensional Lebesgue measure along ∂D. The boundary conditions imply that the boundary integral is nonnegative (This is a simple case of a Darrozes-Giraud inequality, see [6]). Also the term on the right hand side is bounded from above (independently of r) by $\frac{1}{4} \int_D (B(T_1) - B(T_2))^2 dx$. Since T_1 and T_2 take values in the bounded interval $[\gamma_1, \gamma_2]$, the function B is Lipschitz continuous and we obtain the existence of a constant c_1 such that

$$\| \langle r \rangle \|^2_{L^\infty(0,\tau;L^2(D))} \le c_1 \tau \| T_1 - T_2 \|^2_{L^\infty(0,\tau;L^2(D))} \,. \tag{14}$$

The function $s = F(T_1) - F(T_2)$ solves the equation

$$\varepsilon^2 \partial_t s = \varepsilon^2 \Delta_x s - B'(\tilde{\theta}) s + \langle r \rangle \,,$$

subject to homogeneous initial and boundary conditions. A straightforward energy estimate is

$$\| s \|^2_{L^\infty(0,\tau;L^2(D))} \le c_2 \tau \| \langle r \rangle \|^2_{L^\infty(0,\tau;L^2(D))} \,. \tag{15}$$

Combination of (14) and (15) shows that for τ small enough F is a contraction with respect to the norm in $L^\infty(0, \tau; L^2(D))$. ∎

The remainder of this section contains rigorous results related to the asymptotics of the previous section. The following preliminary result is a consequence of an entropy estimate. We shall need that the temperature is bounded away from zero:

A5: $\gamma_1 > 0$.

Lemma 1
Let the assumptions A1–A5 hold. Then for the solution $(I_\varepsilon, T_\varepsilon)$ of (1), (2), (4), (5), (6), and for every $\tau > 0$, $T_\varepsilon \in L^2(0,\tau; H^1(D))$ holds, and there exist positive constants c_1 and c_2, independent of ε, such that

$$\int_0^\tau \int_D (\varepsilon^2 |\nabla_x T_\varepsilon|^2 + \langle (I_\varepsilon - \langle I_\varepsilon \rangle)^2 \rangle + (B(T_\varepsilon) - \langle I_\varepsilon \rangle)^2) dx\, dt$$
$$\leq \varepsilon^2 c_1 + \varepsilon c_2 \int_{\partial D} \int_{n\cdot\Omega<0} (1-\rho)(I_B - B(T_B))^2 d\Omega\, d\sigma\,.$$

Proof: Let $T_B \in H^1(D) \cap L^\infty(D)$ denote an extension of the boundary data T_B to the interior (which exists by assumption A3). Then the following steps are carried out: the equation (1) is multiplied by $I_\varepsilon - B(T_B)$ and the average over S^2 is taken. Then (2) is multiplied by $B(T_\varepsilon) - B(T_B)$. The sum of the resulting two equations is integrated over D. After integrating by parts we obtain

$$\begin{aligned}
&\varepsilon^2 \frac{d}{dt} \int_D \left(\frac{M}{2} \langle I_\varepsilon^2 \rangle + H(T_\varepsilon) - B(T_B)(M\langle I_\varepsilon \rangle + T_\varepsilon) \right) dx \\
&+\varepsilon \int_{\partial D} \left(\frac{1}{2} \langle n\cdot\Omega\, I_\varepsilon^2 \rangle - B(T_B)\langle n\cdot\Omega\, I_\varepsilon \rangle \right) d\sigma \\
&+\varepsilon \int_D \nabla_x B(T_B) \cdot \langle \Omega I_\varepsilon \rangle dx = -\varepsilon^2 \int_D \nabla_x (B(T_\varepsilon) - B(T_B)) \cdot \nabla_x T_\varepsilon\, dx \\
&- \int_D (B(T_\varepsilon) - \langle I_\varepsilon \rangle)^2 dx - \int_D (\langle I_\varepsilon^2 \rangle - \langle I_\varepsilon \rangle^2) dx\,,
\end{aligned} \tag{16}$$

with $H(T) = \int_0^T B(\theta)d\theta$. It is easily shown that the integrand in the first integral of (16) is bounded from below by $H(T_B) - B(T_B)T_B - \frac{M}{2}B(T_B)^2$ and, thus, by a constant.

In the second integral in (16), the integrals over S^2 are split in the incoming and the outgoing contributions. Then, the boundary condition (5) is used and by the coordinate change from Ω to $\Omega' = \Omega - 2n(n\cdot\Omega)$ the integrals over the outgoing directions ($n\cdot\Omega > 0$) are transformed to integrals over the incoming directions. As a result, the integrand in the second integral in (16) can be written as

$$\frac{1}{|S^2|} \int_{n\cdot\Omega<0} |n\cdot\Omega| \left(\frac{1}{2} {I'_\varepsilon}^2 - \frac{1}{2}(\rho I'_\varepsilon + (1-\rho)I_B)^2 - B(T_B)(1-\rho)(I'_\varepsilon - I_B) \right) d\Omega\,,$$

where the prime denotes evaluation at Ω'. The term in the parenthesis is bounded from below by

$$-\frac{1}{2}\frac{1-\rho}{1+\rho}(I_B - B(T_B))^2\,.$$

In the first integral on the right hand side of (16), assumption A5 is used to show that the integrand is bounded from below by

$$\frac{B'(\gamma_1)}{2} |\nabla_x T_\varepsilon|^2 - \frac{1}{2B'(\gamma_1)} |\nabla_x B(T_B)|^2\,.$$

Finally, we combine the last integrals of the left and the right hand sides of (16). Introducing $I_\varepsilon^\perp = I_\varepsilon - \langle I_\varepsilon\rangle$, we obtain for the integrand

$$\begin{aligned}&\langle I_\varepsilon^2\rangle - \langle I_\varepsilon\rangle^2 + \varepsilon\nabla_x B(T_B)\cdot\langle\Omega I_\varepsilon\rangle = \langle (I_\varepsilon^\perp)^2\rangle + \varepsilon\nabla_x B(T_B)\cdot\langle\Omega I_\varepsilon^\perp\rangle \\ &\geq \frac{1}{2}\langle (I_\varepsilon^\perp)^2\rangle - \frac{\varepsilon^2}{2}|\nabla_x B(T_B)|^2 .\end{aligned}$$

Integrating (16) with respect to t and combining the above estimates completes the proof. ■

The lemma shows already that $B(T_\varepsilon)$ and I_ε tend to the same limit. However, so far our estimates are not strong enough. In particular, we can only conclude weak convergence of I_ε and T_ε from the uniform L^∞-estimate of theorem 1. However, strong convergence is needed for going to the limit in the nonlinear term $B(T_\varepsilon)$. A convergence result will be shown in the following under additional assumptions on the initial and boundary data:

A6: $(1-\rho)(I_B - B(T_B)) = 0$ almost everywhere on $\partial D\times S^-$,
$T_I \in W^{2,\infty}(D)$ and $I_I = B(T_I) - \varepsilon\Omega\cdot\nabla_x B(T_I)$.

The analysis of the Milne problem below shows that the assumption on the boundary data excludes boundary layers. With lemma 1 it implies $I_\varepsilon - B(T_\varepsilon) = O(\varepsilon)$. The assumption on the initial data implies uniform boundedness of the time derivatives at $t=0$:

$$M\partial_t I_\varepsilon\,|_{t=0} = \sum_{i,j=1}^{3}\Omega_i\Omega_j\frac{\partial^2 B(T_I)}{\partial x_i\partial x_j}\,, \qquad \partial_t T_\varepsilon\,|_{t=0} = \Delta_x T_I \tag{17}$$

Theorem 2
Let the asssumptions A1–A6 hold. Then, for every $\tau > 0$, the solution $(I_\varepsilon, T_\varepsilon)$ of problem (1), (2), (4), (5), (6) converges as $\varepsilon\to 0$ to $(B(T_0), T_0)$. The convergence of T_ε is in $H^1(D\times(0,\tau))$ weak and the convergence of I_ε in $L^\infty(D\times S^2\times(0,\tau))$ weak*. The limiting temperature T_0 is the unique solution of the problem

$$\begin{aligned}&\partial_t(MB(T_0) + T_0) = \Delta_x(T_0 + B(T_0)/3)\,,\\ &T_0(x,0) = T_I(x)\,, \qquad T_0(\hat{x},t) = T_B(\hat{x})\,, \quad \text{for } \hat{x}\in\partial D\,.\end{aligned}$$

Proof: The bounds in theorem 1 are uniform with respect to ε. Therefore we have convergence of I_ε to I_0 in $L^\infty(D\times S^2\times(0,\tau))$ weak* and of T_ε to T_0 in $L^\infty(D\times(0,\tau))$ weak* (for a suitably chosen subsequence). Also, assumption A6 and lemma 1 imply that

$$R_\varepsilon = \frac{I_\varepsilon - B(T_\varepsilon)}{\varepsilon}$$

is bounded in $L^2(D\times S^2\times(0,\tau))$ uniformly in ε. This implies weak convergence (again for a subsequence) of R_ε to R_0. Now the transport equation (1) is divided by ε. The limit as $\varepsilon\to 0$ in the distributional sense gives

$$\Omega\cdot\nabla_x I_0 = -R_0\,.$$

On the other hand, the conservation equation (7) can be written as

$$\partial_t(M\langle I_\varepsilon\rangle + T_\varepsilon) + \nabla_x\cdot(\langle\Omega R_\varepsilon\rangle - \nabla_x T_\varepsilon) = 0\,.$$

The limit $\varepsilon\to 0$ in the distributional sense gives

$$\partial_t(M\langle I_0\rangle + T_0) - \nabla_x\cdot(\langle\Omega(\Omega\cdot\nabla_x I_0)\rangle + \nabla_x T_0) = 0\,.$$

From lemma 1 it is also clear that I_0 is equal to the weak limit of $B(T_\varepsilon)$. To conclude that this limit is equal to $B(T_0)$ we need strong convergence of the temperature. Uniform boundedness of spatial derivatives of T_ε is a consequence of assumption A6 and of lemma 1. An estimate for the time derivative will be obtained by considering the linear problem for $(\partial_t I_\varepsilon, \partial_t T_\varepsilon)$ obtained by taking the derivative with respect to time of the equations (1), (2) and of the boundary conditions (5), (6), subject to the initial conditions (17). The resulting problem is a linearized version of the original problem with uniformly bounded initial data and homogeneous boundary conditions. By proceeding analogously to the proof of theorem 1, uniform L^∞ bounds for $\partial_t I_\varepsilon$ and $\partial_t T_\varepsilon$ are obtained. Combined with the bound on the spatial derivatives of T_ε this implies that T_ε is uniformly bounded in $H^1(D\times(0,\tau))$. Strong convergence of T_ε to T_0 and, thus, of $B(T_\varepsilon)$ to $B(T_0)=I_0$ in $L^2(D\times(0,\tau))$ is the consequence. This completes the rigorous derivation of the diffusion equation. The weak H^1-convergence also allows to go to the limit in the initial and boundary conditions for the temperature. ■

We conclude this section with two results for the Milne problem:

Theorem 3
Let the assumptions A1–A5 hold. Then there exists a bounded solution (I,T) of the Milne problem (10), (11).

Proof: Supplementing n by the tangential vectors t_1 and t_2 to obtain an orthogonal basis of $\mathbb{R}^3$, we parametrize S^2 by $\mu\in[-1,1]$ and $\varphi\in[0,2\pi)$ via $\Omega(\mu,\varphi) = -\mu n + \sqrt{1-\mu^2}(t_1\cos\varphi + t_2\sin\varphi)$. Now averages of functions on S^2 with respect to φ are introduced:

$$\bar I(y,\mu) = \frac{1}{2\pi}\int_0^{2\pi} I(y,\Omega(\mu,\varphi))d\varphi\,.$$

A simple computation shows that for a solution (I,T) of (10), (11), $(\bar I,T)$ solves the averaged problem

$$\begin{aligned}\mu\partial_y \bar I &= B(T) - \bar I\,,\\ \partial_y^2 T &= B(T) - \langle\bar I\rangle\,,\end{aligned}\qquad(18)$$

subject to the boundary conditions

$$
\begin{aligned}
T(0) &= T_B(\hat{x}) \,, \\
\bar{I}(0,\mu) &= \rho(\mu)\bar{I}(0,-\mu) + [1-\rho(\mu)]\bar{I}_B(\hat{x},\mu) \,, \quad \mu > 0 \,.
\end{aligned} \tag{19}
$$

The remainder can be computed explicitly:

$$
I - \bar{I} = \begin{cases} (1-\rho)(I_B - \bar{I}_B)e^{-y/\mu} \,, & \mu > 0 \,, \\ 0 \,, & \mu < 0 \,. \end{cases}
$$

Now (18), (19) is replaced by a problem on a bounded interval $[0, Y]$:

$$
\begin{aligned}
&\mu\partial_y J = B(\theta) - J \,, \\
&J(0,\mu) = \rho(\mu)J(0,-\mu) + [1-\rho(\mu)]\bar{I}_B(\mu) \,, \quad \mu > 0 \,, \\
&J(Y,\mu) = J(Y,-\mu) \,, \quad \mu < 0 \,,
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
&\partial_y^2\theta = B(\theta) - \langle J \rangle \,, \\
&\theta(0) = T_B \,, \qquad \partial_y\theta(Y) = 0 \,.
\end{aligned} \tag{21}
$$

Now we construct a fixed point operator $\theta \mapsto F(\theta)$ by first solving (20) for given $\gamma_1 \le \theta \le \gamma_2$. This can be done explicitly, and the estimate $B(\gamma_1) \le J \le B(\gamma_2)$ follows easily. Then θ is replaced by $F(\theta)$ in (21). It is a standard result for semilinear elliptic problems that $F(\theta)$ is uniquely determined and a consequence of the maximum principle that $\gamma_1 \le F(\theta) \le \gamma_2$ holds. As a consequence of the differential equation in (21), F is a compact operator on a bounded set in $L^\infty((0,Y))$. Continuity of F is straightforward. Schauder's fixed point theorem implies existence of a solution of (20), (21).

The above L^∞-estimates are independent of Y. Therefore, as $Y \to \infty$, the restriction of the solution (J, θ) of (20), (21) to a fixed bounded interval $[0, Y_0]$ lies in a bounded subset of $L^\infty((0,Y_0) \times (-1,1)) \times W^{2,\infty}((0,Y_0))$. This is sufficient for carrying out the limit $Y \to \infty$, which completes the proof. ∎

The standard Milne problem of radiative transfer (or neutron transport) [3, 18] does not contain the heat conduction term $\partial_y^2 T$ in (10) and reduces therefore to a linear kinetic equation. Standard results include vanishing flux for bounded solutions and that an energy integral is conserved across the boundary layer. These results carry over to the problem considered here: combining the two equations in (10) gives the energy conservation equation $\partial_y(\langle \mu I \rangle - \partial_y T) = 0$ and, thus, the total energy flux is constant:

$$
\langle \mu I \rangle - \partial_y T = j \,.
$$

Multiplication of the first equation in (10) by μ and averaging gives $\partial_y(T + \langle \mu^2 I \rangle) = -j$. Boundedness of the solution implies

$$
j = 0 \,, \qquad T + \langle \mu^2 I \rangle = K \,.
$$

Finally, we show that the assumption A6 removes the boundary layer:

Lemma 2
Let the assumptions A1–A6 hold. Then the solution of the Milne problem constructed in theorem 3 is given by

$$I(y,\Omega) = B(T_B)\,, \qquad T(y) = T_B\,.$$

Proof: With the notation of the proof of theorem 3, the assumption A6 obviously implies $I - \bar{I} = 0$. In the problem (20), (21) on the bounded interval, we proceed similarly to the proof of lemma 1: the differential equation in (20) is multiplied by $B(T_B) - J$ and averaged with respect to μ. Then the differential equation in (21) is multiplied by $B(\theta) - B(T_B)$. The two equations are added and integrated from $y = 0$ to Y. The result can be written as

$$\int_0^Y \Big(B'(\theta)(\partial_y \theta)^2 + (\langle J\rangle - B(\theta))^2 + \langle (J - \langle J\rangle)^2\rangle \Big)\, dy$$
$$= \langle \mu J(J/2 - B(T_B))\rangle \Big|_{y=0} \leq \int_0^1 \frac{\mu}{4}\, \frac{1-\rho}{1+\rho} \Big(\bar{I}_B - B(T_B) \Big)^2 d\mu\,,$$

where the last inequality is proven as in the proof of lemma 1. The term on the right hand side vanishes by assumption A6, implying $J = B(\theta) = \text{const.}$ Passing to the limit $Y \to \infty$ completes the proof. ∎

4 A Numerical Procedure Induced by the Asymptotic Limit

For small values of ε, problem (1), (2) is stiff. In this section we are interested in developing a numerical scheme which is suitable for simulations of the equations in the small mean free path limit. The algorithm should work uniformly for all ε and tend to a suitable scheme for the diffusion equation as ε tends to 0. In particular, it should not be necessary to adapt the stepsize as $\varepsilon \to 0$. These points are obviously not satisfied for a straightforward explicit discretization of the equations. However, they can be achieved by a semi-implicit discretization. For $\varepsilon \to 0$ the discretization presented below tends to a standard linear implicit method for the nonlinear diffusion equation (8). To obtain a suitable discretization of (1), (2) we start by rewriting the problem.

The asymptotic procedure of section 2 suggests writing the radiative intensity in the form

$$I(x,\Omega,t) = B(T(x,t)) + \varepsilon \hat{I}(x,\Omega,t) + \varepsilon^2 Z(x,t)\,, \tag{22}$$

with $\langle \hat{I}\rangle = 0$ (we skip the subscript ε from now on). This is a decomposition of I into its mean value $\langle I\rangle = B(T) + \varepsilon^2 Z$ with respect to Ω and the remainder $\varepsilon \hat{I}$. Instead of I, we use the new unknowns $\hat{I}$ and Z.

The equations (1), (2) are now rewritten as a system for $\hat{I}$, Z and T by substituting (22), taking the mean value of (1), and subtracting it from (1):

$$\varepsilon^2 M\partial_t Z + MB'(T)(\Delta_x T + Z) + \nabla_x \cdot \langle \Omega\hat{I}\rangle = -Z\,, \tag{23}$$
$$\varepsilon^2 M\partial_t \hat{I} + \Omega\cdot\nabla_x B(T) + \varepsilon\nabla_x\cdot(\Omega\hat{I} - \langle\Omega\hat{I}\rangle) + \varepsilon^2\nabla_x\cdot(\Omega Z) = -\hat{I}\,, \tag{24}$$
$$\partial_t T = \Delta_x T + Z\,. \tag{25}$$

From the initial conditions (4) we obtain

$$\hat{I}(x,\Omega,0) = \frac{1}{\varepsilon}[I_I(x,\Omega) - \langle I_I\rangle(x)]\,, \qquad Z(x,0) = \frac{1}{\varepsilon^2}[\langle I_I\rangle(x) - B(T_I(x))]\,,$$
$$T(x,0) = T_I(x)\,. \tag{26}$$

The transformed system consists of the ordinary differential equation (23) for Z (for given $\hat{I}$ and T), the kinetic equation (24) for $\hat{I}$, and the parabolic equation (25) for T. Therefore, boundary conditions for $\hat{I}$ and T are needed.

Using the boundary conditions (5), (6) in the condition $\langle\hat{I}\rangle = 0$, evaluated at the boundary ∂D, we obtain the equation

$$\int_{n\cdot\Omega>0}(1+\rho(-n\cdot\Omega))\hat{I}d\Omega + \int_{n\cdot\Omega<0}(1-\rho)\left(\frac{I_B - B(T_B)}{\varepsilon} - \varepsilon Z\right)d\Omega = 0\,, \tag{27}$$

on ∂D. As long as the boundary is not purely reflective, i.e.,

$$\int_{n\cdot\Omega<0}(1-\rho)d\Omega > 0\,,$$

the value of Z can be computed from (27) in terms of the outflow data $\hat{I}(\hat{x},\Omega,t)$, $n\cdot\Omega > 0$. Then, boundary conditions for $\hat{I}$ and T are given by

$$\begin{aligned}\hat{I}(\hat{x},\Omega,t) &= \rho(n\cdot\Omega)\hat{I}(\hat{x},\Omega',t)\\ &\quad +(1-\rho(n\cdot\Omega))\left(\frac{I_B(\hat{x},\Omega) - B(T_B(\hat{x}))}{\varepsilon} - \varepsilon Z(\hat{x},t)\right)\,, \end{aligned}\tag{28}$$
$$T(\hat{x},t) = T_B(\hat{x})\,, \tag{29}$$

for $\hat{x}\in\partial D$, $n\cdot\Omega < 0$. Obviously, for a purely reflective boundary, $\rho(n\cdot\Omega) = 1$ holds Ω-a.e., and the value of Z at the boundary is not needed.

In the following the problem in the form (23)–(29) will be discretized.

4.1 Time Discretization

We introduce a time step $\tau > 0$ and approximations of the solution at time $t^n = n\tau$, denoted by the superscript n. We use a semi-implicit discretization of (23)–(25) where backward differences are used for the zeroth order terms (as $\varepsilon \to 0$) and forward differences for higher order terms:

$$
\begin{aligned}
&\frac{\varepsilon^2 M}{\tau}(Z^{n+1} - Z^n) + MB'(T^n)(\Delta_x T^{n+1} + Z^{n+1}) \\
&\qquad\qquad +\nabla_x \cdot \langle \Omega \hat{I}^{n+1} \rangle = -Z^{n+1} , && (30) \\
&\frac{\varepsilon^2 M}{\tau}(\hat{I}^{n+1} - \hat{I}^n) + B'(T^n)\Omega \cdot \nabla_x T^{n+1} + \varepsilon \nabla_x \cdot (\Omega \hat{I}^n - \langle \Omega \hat{I}^n \rangle) \\
&\qquad\qquad +\varepsilon^2 \nabla_x \cdot (\Omega Z^n) = -\hat{I}^{n+1} , && (31) \\
&\frac{1}{\tau}(T^{n+1} - T^n) = \Delta_x T^{n+1} + Z^{n+1} . && (32)
\end{aligned}
$$

Note that linearization about the old time step has been used to make the problem at the new time step linear. From the equations (30), (31), Z^{n+1} and $\hat{I}^{n+1}$ can be computed explicitly in terms of T^{n+1}. Using the result in (32) and setting $\kappa = 1 + \varepsilon^2 M/\tau$ gives

$$
\begin{aligned}
\hat{I}^{n+1} &= -\frac{1}{\kappa} B'(T^n)\Omega \cdot \nabla_x T^{n+1} + \varepsilon G^n , && (33) \\
&\quad \text{with } G^n = \frac{1}{\kappa}\left(\frac{\varepsilon M}{\tau}\hat{I}^n - \nabla_x \cdot (\Omega \hat{I}^n - \langle \Omega \hat{I}^n \rangle) - \varepsilon \nabla_x \cdot (\Omega Z^n) \right) , \\
Z^{n+1} &= (\kappa + MB'(T^n))^{-1} \left[\frac{1}{\kappa} \nabla_x \cdot \left(\frac{1}{3} B'(T^n) \nabla_x T^{n+1} \right) \right. \\
&\qquad\qquad \left. -MB'(T^n)\Delta_x T^{n+1} + \varepsilon H^n \right] , && (34) \\
&\quad \text{with } H^n = \frac{\varepsilon M}{\tau} Z^n - \nabla_x \cdot \langle \Omega G^n \rangle , \\
(\kappa &+ MB'(T^n)) \frac{1}{\tau}(T^{n+1} - T^n) \\
&= \nabla_x \cdot \left[\left(\kappa + \frac{1}{3\kappa} B'(T^n) \right) \nabla_x T^{n+1} \right] + \varepsilon H^n . && (35)
\end{aligned}
$$

Thus, an elliptic equation for T^{n+1} remains to be solved. For $\varepsilon = 0$, the last equation becomes a stable, linear-implicit discretization of the diffusion equation (9).

4.2 Space Discretization

To state the full discretization we consider for simplicity the one-dimensional case $x \in [0, 1]$, where we also replace Ω by the one-dimensional variable $\mu \in [-1, 1]$. Thus, averaging is now understood with respect to μ: $\langle f \rangle = \frac{1}{2} \int_{-1}^{1} f(\mu) d\mu$

We discretize space using staggered grids with $h = 1/\text{imax}$:

$$x_i = ih, \qquad i = 0, \ldots, \text{imax} ,$$

and

$$x_{i-1/2} = (i - 1/2)h, \qquad i = 0, \ldots, \text{imax} + 1 .$$

The variables T, and Z are discretized at the full grid points x_i, and $\hat{I}$ is discretized at the points $x_{i-1/2}$. The approximations at time t^n are denoted by T_i^n, Z_i^n, and $\hat{I}_{i-1/2}^n$. Defining

$$(B')_i^n = B'(T_i^n)\,, \qquad (B')_{i-1/2}^n = \frac{1}{2}\left(B'(T_i^n) + B'(T_{i-1}^n)\right)\,,$$

the one-dimensional, space-discretized version of (33)–(35) reads

$$\hat{I}_{i-1/2}^{n+1} = -\frac{1}{\kappa}(B')_{i-1/2}^n \mu \frac{T_i^{n+1} - T_{i-1}^{n+1}}{h} + \varepsilon G_{i-1/2}^n\,, \tag{36}$$

$$\text{with } G_{i-1/2}^n = \frac{1}{\kappa}\Bigg(\frac{\varepsilon M}{\tau}\hat{I}_{i-1/2}^n - \frac{\mu}{h}(\hat{I}_i^n - \hat{I}_{i-1}^n) + \left\langle \frac{\mu}{h}(\hat{I}_i^n - \hat{I}_{i-1}^n)\right\rangle) - \frac{\varepsilon\mu}{h}(Z_i^n - Z_{i-1}^n)\Bigg)\,, \tag{37}$$

$$Z_i^{n+1} = \left(\kappa + M(B')_i^n\right)^{-1}\Bigg[-\frac{M}{h^2}(B')_i^n(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}) + \varepsilon H_i^n + \frac{1}{3\kappa h}\left((B')_{i+1/2}^n\frac{T_{i+1}^{n+1} - T_i^{n+1}}{h} - (B')_{i-1/2}^n\frac{T_i^{n+1} - T_{i-1}^{n+1}}{h}\right)\Bigg]\,, \tag{38}$$

$$\text{with } H_i^n = \frac{\varepsilon M}{\tau}Z_i^n - \left\langle\frac{\mu}{h}(G_{i+1/2}^n - G_{i-1/2}^n)\right\rangle\,, \tag{39}$$

$$(\kappa + M(B')_i^n)\frac{1}{\tau}(T_i^{n+1} - T_i^n) = \frac{1}{h}\Bigg[\left(\kappa + \frac{1}{3\kappa}(B')_{i+1/2}^n\right)\frac{T_{i+1}^{n+1} - T_i^{n+1}}{h} - \left(\kappa + \frac{1}{3\kappa}(B')_{i-1/2}^n\right)\frac{T_i^{n+1} - T_{i-1}^{n+1}}{h}\Bigg] + \varepsilon H_i^n\,. \tag{40}$$

The free streaming operator in (37) is discretized by an upwinding procedure:

$$\hat{I}_i^n = \begin{cases} \hat{I}_{i-1/2}^n\,, & \text{for } \mu > 0\,, \\ \hat{I}_{i+1/2}^n\,, & \text{for } \mu < 0\,. \end{cases}$$

The comments about the system (33)–(35) carry over to (36)–(40). In particular, for $\varepsilon = 0$, (40) becomes a standard linear-implicit discretization of (9):

$$\begin{aligned}&(1 + M(B')_i^n)\frac{h^2}{\tau}(T_i^{n+1} - T_i^n) \\ &= (1 + (B')_{i+1/2}^n/3)(T_{i+1}^{n+1} - T_i^{n+1}) - (1 + (B')_{i-1/2}^n/3)(T_i^{n+1} - T_{i-1}^{n+1})\,.\end{aligned}$$

It remains to discretize the boundary conditions. We connect their presentation with the presentation of the algorithm which needs to be carried out for each time step:

Step 1: Compute $G_{i-1/2}^n$, $i = 1, \ldots$,imax, from (37).

Step 2: Compute H_i^n, $i = 1, \ldots$,imax-1, from (39).

Step 3: Compute T_i^{n+1}, $i = 0, \ldots$,imax, from (40) subject to the boundary conditions $T_0^{n+1} = T_B(0)$, $T_{\text{imax}}^{n+1} = T_B(1)$.

Step 4: Compute Z_i^{n+1}, $i = 1, \ldots$,imax-1, from (38).

Step 5: Compute $\hat{I}_{i-1/2}^{n+1}$, $i = 1, \ldots$,imax, from (36).

Step 6: Finally, $\hat{I}_{-1/2}^{n+1}$ for $\mu > 0$ ($\hat{I}_{\text{imax}+1/2}^{n+1}$ for $\mu < 0$) is determined from the boundary conditions (27) and (28) in the following way: in (27), the outflow data for $\hat{I}$ are approximated by $\hat{I}_{1/2}^{n+1}$, $\mu < 0$ ($\hat{I}_{\text{imax}-1/2}^{n+1}$, $\mu > 0$). The equation (27) at $x = 0$ ($x = 1$) then determines Z_0^{n+1} (Z_{imax}^{n+1}) (if the respective boundary point is not purely reflective). This value is then used in the right hand side of (28), where the outflow data on the right hand side are again approximated like above and the left hand side is replaced by $(\hat{I}_{-1/2}^{n+1} + \hat{I}_{1/2}^{n+1})/2$ $((\hat{I}_{\text{imax}-1/2}^{n+1} + \hat{I}_{\text{imax}+1/2}^{n+1})/2)$. The values of $\hat{I}_{-1/2}^{n+1}$, $\mu > 0$, and $\hat{I}_{\text{imax}+1/2}^{n+1}$, $\mu < 0$, computed from these equations are needed in Step 1 of the following time step.

Note that only Step 3 is implicit. A tridiagonal system resulting from the implicit discretization of a parabolic equation needs to be solved there.

4.3 Velocity Discretization

For the discretization of the μ-variable a discrete velocity model is used, where the discrete velocities are the zeroes of a fixed Legendre polynomial. The averages in (37) and (39) are then computed by Gaussian integration. Note that a different interpretation of this method is possible: it can be obtained as a Galerkin method, where $\hat{I}$ (as a function of μ) is expanded in Legendre polynomials (see [20] for the connection between the two interpretations in the context of semiconductor modelling).

5 Numerical Results

We consider the one-dimensional slab-problem introduced in the previous section with the parameter values $M = 1$, $\sigma = 1$ (in (3)). The boundary conditions for the temperature are given by

$$T_B(0) = 1\,, \qquad T_B(1) = 0\,. \tag{41}$$

We further assume a nonreflecting boundary, i.e., $\rho = 0$, and two sets of inflow data: *equilibrium boundary conditions* (satisfying assumption A6):

$$I_B(0,\mu) = 1\,,\ \mu > 0\,, \qquad I_B(1,\mu) = 0\,,\ \mu < 0\,, \tag{42}$$

or *nonequilibrium boundary conditions* (at $x = 0$):

$$I_B(0,\mu) = \mu\,,\;\; \mu > 0\,, \qquad I_B(1,\mu) = 0\,,\;\; \mu < 0\,. \tag{43}$$

The initial data are $I_I = T_I = 0$.

For the velocity discretization 16 grid points are used. The spatial grid spacing is $h = 0.01$. Experiments with different values of ε have been carried out. According to rough stability considerations (similar to those in [21]), a CFL-condition of the form $\tau \le c\varepsilon h$ has to be satisfied by the time step, if h is of the order of ε or smaller. If, on the other hand, h is large compared to ε, unconditional stability is to be expected. Thus, the strongest time step restriction applies for the largest value of ε, we intend to compute with. We always used the time step $\tau = 0.001$, leading to a stable scheme for $\varepsilon \le 0.1$. Note that, for small values of ε, stability can be achieved with a coarse grid. More precisely, the grid can be chosen independently of ε for $\varepsilon \to 0$.

We present results illustrating that the numerical scheme is able to reproduce the asymptotic analysis of Sections 2 and 3. Note that for steady states of the limiting one-dimensional diffusion model (9) the quantity $T_0 + B(T_0)/3$ is a linear function of position. We therefore plot this quantity,

$$T + T^4/3\,,$$

in the following figures. As a reference, all figures contain the stationary solution of the diffusion equation subject to the boundary data (41).

The computations are carried out long enough for steady states to be established numerically. In Figure 1, the results at several time steps for $\varepsilon = 0.001$ and equilibrium boundary conditions (42) are depicted. The steady state is graphically indistinguishable from the corresponding solution of the diffusion equation (9). Figures 2 and 3 show the corresponding results for nonequilibrium boundary conditions (43), and $\varepsilon = 0.1$ and $\varepsilon = 0.001$, respectively. The effect of the kinetic boundary layer is clearly visible. The correct boundary condition at $x = 0$ for the limiting diffusion equation is not given by (41). It requires the solution of a Milne problem (see section 3). Finally, the convergence behaviour as $\varepsilon \to 0$ of stationary solutions for nonequilibrium boundary conditions is demonstrated in Figure 4.

6 Conclusions

- A theoretical investigation of radiative heat transfer, nonlinearly coupled with heat conduction has been performed. The convergence to the diffusion equation for small Knudsen numbers and the existence of solutions of the Milne problem have been proven rigorously.

- A numerical procedure which preserves the asymptotics leading to the nonlinear diffusion limit is presented. It allows to use coarse grids (in particular time steps) for small Knudsen numbers.

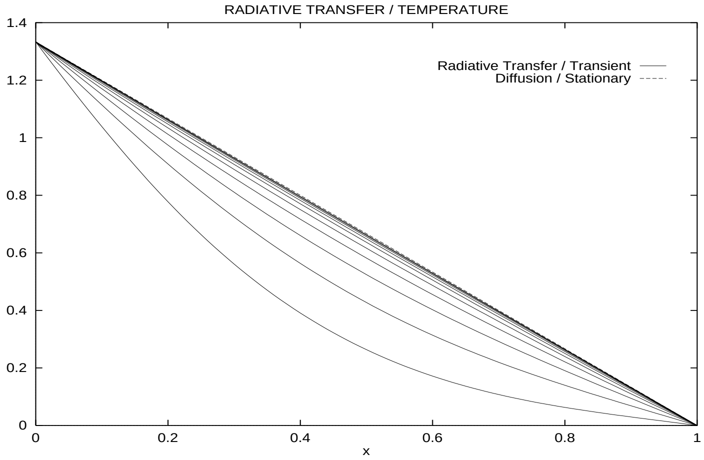

Figure 1: $\varepsilon = 0.001$

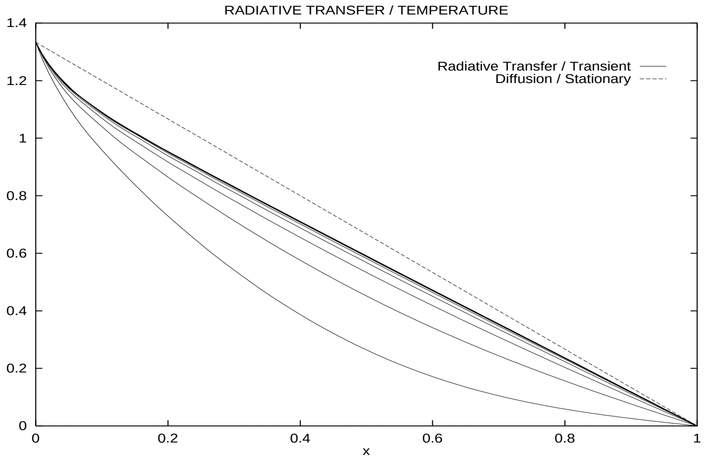

Figure 2: $\varepsilon = 0.1$

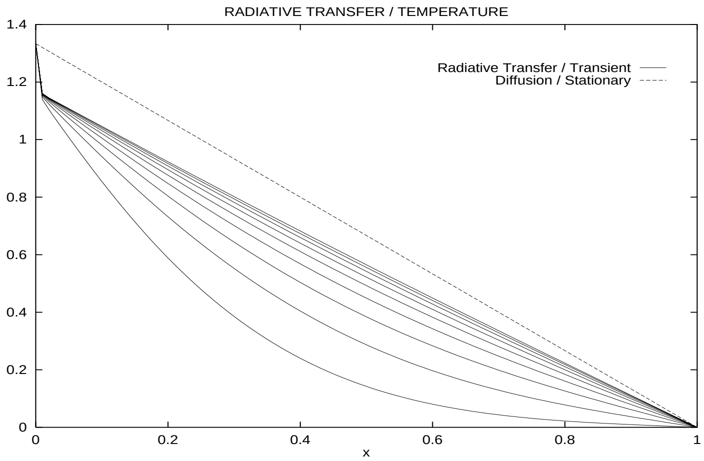

Figure 3: $\varepsilon = 0.001$

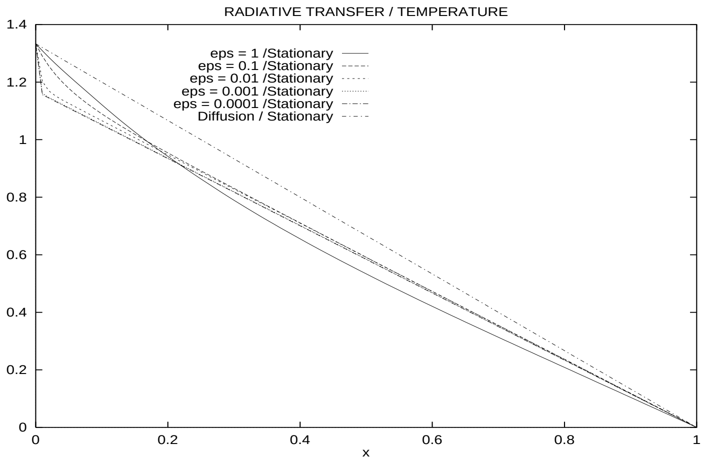

Figure 4: Stationary solutions

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