Lecture Notes on

Pattern Formation

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1 Introduction

What is a *pattern?* In a scientific context this term usually has a more general meaning than in colloquial English. It stands for any form of recognizable *order*. This last expression in turn has a probabilistic defintion which distinguishes it from its opposite, *chaos*. We only state an example: If, in the room of a teenager, the probability to find clothes in the closet is at least marginally higher than the probability to find them in any other place, this already constitutes a form of order, although the average parent might not agree with this defintion.

In this course we shall be interested in understanding mechanisms for the creation of patterns, and our motivation will be taken from biology. Some of the most fascinating and still poorly understood pattern formation mechanisms occur in the development of embryos. The wider term *morphogenesis*, i.e. the creation of different forms and shapes, is often used in this context. Meta-theories often state the existence of *morphogenes* as the carriers of structural information. Pattern formation is then described as a process, where many morphogenes of different types interact with each other and with the environment, influencing their creation, annihilation, and movement.

More generally, we shall consider large ensembles of what we call *particles*. This expression is borrowed from physics. Here, a particle will be any object able to move individually, and which is small compared to the length scales we are interested in. This leads to the idealization of *point particles*. Depending on the situation these can be molecules, cells, or even multicellular organisms. As a first step, we shall derive mathematical models for the movement of large particle ensembles.

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2 Random motion of particles

Biological particles usually live in a complex nonhomogeneous environment influencing their movement. As a consequence, for an observer this movement looks like having a random component. We therefore accept a random nature of this movement as a postulate for a mathematical description. Another postulate, which makes life much easier (although it is not justified in general), is that regarding the random component of motion the particles are independent in the probabilistic sense.

We start by considering a discrete one-dimensional random motion. Let $x_j = j\Delta x$, $j \in \mathbb{Z}$, denote the possible positions of particles, and assume that at the discrete points $t_n = n\Delta t$, $n \in \mathbb{Z}$, in time particles perform jumps of the length Δx to the left or to the right. Let us assume further that the probability of jumping to the left is q, and the probability of jumping to the right is $1-q$ (with $0 \le q \le 1$, of course). Now we introduce the nonnegative quantities p_j^n, $j, n \in \mathbb{Z}$, which can be interpreted either as the probability that one particle is at the position x_j at time t_n or as the expected number of particles out of a large ensemble at position x_j at time t_n or (if the latter is divided by Δx) as the expected number density of particles at position x_j at time t_n. Then, obviously the values at time t_{n+1} can be computed in terms of the values at time t_n:

$$p_j^{n+1} = qp_{j+1}^n + (1-q)p_{j-1}^n \tag{1}$$

Eventually we are looking for continuous descriptions both in time and in position. Therefore we shall interpret p_j^n as approximation for $p(x_j, t_n)$ where p is a function of two real valued arguments. With this interpretation in mind we rewrite the above equation as

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} - \frac{q\Delta x}{\Delta t}\,\frac{p_{j+1}^n - p_j^n}{\Delta x} + \frac{(1-q)\Delta x}{\Delta t}\,\frac{p_j^n - p_{j-1}^n}{\Delta x} = 0\,.$$

Our aim is to pass to the limit $\Delta x, \Delta t \to 0$. Obviously the result depends on the relative size of Δx and Δt. We have three main options: Either the grid speed $s := \Delta x/\Delta t$ tends to zero, to infinity, or we keep it fixed at a positive finite value. The most interesting result occurs in the latter case, which we call the *significant limit*:

$$\partial_t p + \partial_x(vp) = 0\,, \qquad \text{with } v = s(1-2q)\,. \tag{2}$$

Actually, the other two cases can be recovered by letting $s \to 0$ or $s \to \infty$.

Equation (2) is a one-dimensional *convection equation*. Solutions are travelling waves $p(x,t) = f(x - vt)$ with velocity v. With the interpretation of p as time dependent density of particles along the line, the integrated version of (2),

$$\frac{d}{dt}\int_a^b p(x,t)dx + vp(b,t) - vp(a,t) = 0\,,$$

gives the rate of change of the number of particles contained in the interval (a,b). The term $j(x,t) = vp(x,t)$ can then be interpreted as the *flux* of particles through the point x at time t, and v is the mean velocity of particles.

It is interesting to note that equation (2) could have been derived without any probabilistic effects. The assumption that all particles always move to the right or always to the left, i.e., $q = 0$ or $q = 1$, still leads to (2) with $v = \pm s$. More generally, the same value of v, and therefore the same macroscopic equation (2) can be obtained by different choices of the grid speed s and of the probability q. This shows that the properties of the microscopic movement cannot be completely recovered from macroscopic observations.

In the symmetric situation $q = 1/2$, the mean velocity vanishes, and (2) becomes trivial. This unsatisfactory situation can be clarified by returning to the discrete equation (1) and by rewriting it in a different way:

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} - \frac{(\Delta x)^2}{2\Delta t}\,\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{(\Delta x)^2} = 0\,.$$

This shows that for $q = 1/2$, the significant limit is achieved, when $D = (\Delta x)^2/(2\Delta t)$ is kept fixed as $\Delta x, \Delta t \to 0$:

$$\partial_t p - D\partial_x^2 p = 0\,. \tag{3}$$

This is the one-dimensional *diffusion equation* with diffusivity D. Integration as above shows that the diffusive flux is given by *Fick's law* $j = -D\partial_x p$.

As for the convection equation we want to demonstrate that the diffusion equation can also be obtained as macroscopic model for different microscopic dynamics as long as they do not have a directional bias. Instead of a *position jump process* as considered above, we now describe a *velocity jump process*. Consider particles, which move along the line with velocity $s > 0$ or $-s$. At discrete points $t_n = n\Delta t$ in time they change to the other velocity with probability q. We denote the expected density of particles moving to the right at time t_n (after the velocity jump) by $r^n(x)$, and the expected density

of particles moving to the left at time t_n by $l^n(x)$. Then the densities at time t_{n+1} before the velocity jump are given by

$$\hat{r}^{n+1}(x) = r^n(x - s\Delta t)\,, \qquad \hat{l}^{n+1}(x) = l^n(x + s\Delta t)\,.$$

After the velocity jump at time t_{n+1} we obtain

$$r^{n+1}(x) = (1-q)\hat{r}^{n+1}(x) + q\hat{l}^{n+1}(x)\,, \qquad l^{n+1}(x) = (1-q)\hat{l}^{n+1}(x) + q\hat{r}^{n+1}(x)\,,$$

which can be rewritten as

$$\frac{r^{n+1}(x) - r^n(x)}{\Delta t} + s\frac{r^n(x) - r^n(x - s\Delta t)}{s\Delta t} = \frac{q}{\Delta t}(l^n(x + s\Delta t) - r^n(x - s\Delta t))\,,$$
$$\frac{l^{n+1}(x) - l^n(x)}{\Delta t} - s\frac{l^n(x + s\Delta t) - l^n(x)}{s\Delta t} = \frac{q}{\Delta t}(r^n(x - s\Delta t) - l^n(x + s\Delta t))\,.$$

A significant limit is obtained with the scaling assumption that $\tau := \Delta t/q$ remains fixed as $\Delta t \to 0$:

$$\partial_t r + s\partial_x r = \frac{l - r}{\tau}\,, \qquad \partial_t l - s\partial_x l = \frac{r - l}{\tau}\,.$$

This is the simplest example of a *kinetic transport equation*, describing an ensemble of particles not only by its positional distribution but also its distribution with respect to velocity. Kinetic transport equations are often called *mesoscopic models*. In a *macroscopic scaling*, x is replaced by x/ε and t is replaced by t/ε^2, where ε is a small positive dimensionless parameter. This leads to the rescaled version

$$\varepsilon^2\partial_t r + \varepsilon s\partial_x r = \frac{l - r}{\tau}\,, \qquad \varepsilon^2\partial_t l - \varepsilon s\partial_x l = \frac{r - l}{\tau}\,. \tag{4}$$

For carrying out the macroscopic limit $\varepsilon \to 0$, we replace the system by the first equation an the sum of the equations. After dividing by ε and, respectively, by ε^2, we obtain

$$\varepsilon\partial_t r + s\partial_x r = \frac{l - r}{\varepsilon\tau}\,, \qquad \partial_t p + s\partial_x \frac{r - l}{\varepsilon} = 0\,, \tag{5}$$

where $p = r + l$ is the total (or macroscopic) density. In the limit $\varepsilon \to 0$, (4) gives $r = l$, and the first equation in (5) shows that the flux $s(r - l)/\varepsilon$ converges to $-s^2\tau\partial_x r = -D\partial_x p$ with $D = s^2\tau/2$, such that we again obtain the diffusion equation (3).

So far we have seen that the macroscopic limit of a biased random motion is a convection equation, and for an unbiased motion it is a diffusion equation. Actually, both effects can be combined in the macroscopic equation

by an appropriate scaling assumption. We shall also generalize the position jump process by allowing a dependence of the jump probability on position and time:

$$p_j^{n+1} = q_{j+1}^n p_{j+1}^n + (1 - q_{j-1}^n) p_{j-1}^n \,.$$

As in the derivation of the diffusion equation we assume that $D = (\Delta x)^2/(2\Delta t)$ is fixed and that the jump probabilities are close to $1/2$:

$$q_j^n = \frac{1}{2} - \frac{v(x_j, t_n)\Delta t}{2\Delta x} \,,$$

where $v(x,t)$ is a given velocity function. The analogous computations as in the derivation of the diffusion equation now lead to the one-dimensional *convection-diffusion equation*

$$\partial_t p + \partial_x (vp - D\partial_x p) = 0 \,. \tag{6}$$

Everything we did so far can be extended to higher dimensions with the result

$$\partial_t p + \nabla \cdot (vp - D\nabla p) = 0 \,, \tag{7}$$

where now the density $p(x,t)$ depends on position $x \in \mathbb{R}^d$, with $d = 2$ or $d = 3$, and on time $t \in \mathbb{R}$. The gradient with respect to x is denoted by ∇ and the divergence by $\nabla\cdot$. The velocity $v(x,t)$and the flux $vp - D\nabla p$ are vector fields. The interpretation of the flux vector is the following: Its component in the direction ν is the number of particles per time and per unit area moving through an area element orthogonal to ν. This can be seen by integrating (7) over a bounded position domain $\Omega \subset \mathbb{R}^d$ and using the divergence theorem:

$$\frac{d}{dt}\int_\Omega p\, dx + \int_{\partial\Omega} (vp - D\nabla p)\cdot \nu \, d\sigma = 0 \,, \tag{8}$$

where ν denotes the unit outward normal vector along the boundary $\partial\Omega$, and $d\sigma$ is the line element for $d = 2$ and the surface element for $d = 3$.

So far we only described the movement of particles. Equation (7) is a conservation law. No particles are created or destroyed. As the final step in this modelling section, we also allow for this possibility. We denote by $f(x,t)$ the number of particles created or destroyed (depending on the sign of f) per unit time and unit volume. Then the right hand side of (8) has

to be replaced by the integral of f over Ω, and the differential version (7) becomes the *reaction-convection-diffusion equation*

$$\partial_t p + \nabla \cdot (vp - D\nabla p) = f\,. \tag{9}$$

Why reaction? In a typical situation our particles are molecules whose creation or destruction is the result of a chemical reaction. Another interpretation of f in the following will be as a birth/death term.

In the following, systems of equations of the form (9) for different species of particles will be considered, when x varies in a domain $\Omega \subset \mathbb{R}^d$. Typically we shall either assume that Ω is bounded with zero flux boundary conditions

$$(vp - D\nabla p) \cdot \nu = 0 \quad \text{along } \partial\Omega\,,$$

or, as an idealization, that $\Omega = \mathbb{R}^d$. In the latter case it is usually assumed that $f(x,t) \to 0$ as $|x| \to \infty$, and that either the total number of particles is bounded, i.e., $\int_{\mathbb{R}^d} p\,dx < \infty$, or that $p(x,t)$ converges to a constant value as $|x| \to \infty$.

For given v and f, the formulation of a well posed problem for the unknown p is completed by prescribing initial conditions $p(x,0) = p_I(x)$ for $x \in \Omega$, with given initial data p_I. Well posedness means that the initial-boundary value problem has a unique solution continuously dependent on the data v, f, and p_I.

3 Stability of homogeneous steady states

In this section we consider reaction and diffusion of one species of particles in a stationary homogeneous environment, i.e., equations of the form

$$\partial_t p - D\Delta p = f(p)\,, \tag{10}$$

where $\Delta = \nabla\cdot\nabla$ is the Laplace operator and the stationarity and homogeneity of the environment is reflected by the fact that the reaction rate f does not explictly depend on t or x. A *homogeneous steady state* is a constant solution p_0 of (10), implying that p_0 is a zero of f. If (10) is considered on the position domain Ω, then a homogeneous steady state satisfies zero flux boundary conditions ($\nu \cdot \nabla u = 0$ on $\partial\Omega$) for bounded Ω and, obviously, the condition $p(x,t) \to p_0$ for $|x| \to \infty$ for $\Omega = \mathbb{R}^d$. We shall consider these two situations.

The stability of p_0 is examined by introducing the perturbation $u(x,t) = p(x,t) - p_0$ where p is a solution of (10) close to p_0, i.e., u is small. Substitution in (10), Taylor expanding f around $p = p_0$ ($u = 0$), and keeping only

the linear term gives the *linearized equation*

$$\partial_t u - D\Delta u = f'(p_0)u\,,$$

with zero flux boundary conditions for Ω bounded, and with $u \to 0$ as $|x| \to \infty$ for $\Omega = \mathbb{R}^d$.

For bounded Ω, the linearized problem can be solved by separation of variables leading to a representation of solutions in the form

$$u(x,t) = \sum_{j=0}^{\infty} u_j \varphi_j(x) \exp([f'(p_0) + D\mu_j]t)\,,$$

where $\varphi_0 = 1, \varphi_1, \ldots$ are the eigenfunctions of the Laplace operator subject to zero flux boundary conditions and $\mu_0 = 0, \mu_1, \ldots$ are the corresponding eigenvalues, i.e.,

$$\Delta\varphi_j = \mu_j\varphi_j\,, \qquad \nu \cdot \nabla\varphi_j = 0 \text{ on } \partial\Omega\,,$$

for $j = 0, 1, \ldots$ Different solutions are distinguished by the choice of the constants $u_0, u_1, \ldots$ The computation

$$\int_\Omega \varphi_l \Delta\varphi_j \, dx = -\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_l \, dx$$

implies that the Laplace operator with zero flux boundary conditions is symmetric with respect to the scalar product defined by pointwise multiplication and subsequent integration. This has the consequence that all eigenvalues are real and that $\{\varphi_j,\ j \geq 0\}$ can be chosen as an orthonormal sequence. The above formula also implies that the eigenvalues are nonpositive. Actually it can be shown that $\mu_j \to -\infty$ as $j \to \infty$. W.l.o.g. we assume the eigenvalues to be ordered: $\mu_0 \geq \mu_1 \geq \ldots$

A steady state solution is called *stable* if, when starting with an initial condition close to the steady state the solution remains close to the steady state for all times, it is called *asymptotically stable* if, furthermore, such solutions converge to the steady state as time tends to infinity.

The steady state $u = 0$ of the linearized problem is stable iff $f'(p_0) \leq 0$, it is asymptotically stable iff $f'(p_0) < 0$. In these cases we say that the steady state p_0 of the original nonlinear equation is *linearized (asymptotically) stable*. It can be shown that linearized asymptotic stability implies asymptotic stability and that linearized instability implies instability.

In the case $\Omega = \mathbb{R}^d$, we solve the linearized equation by the *Fourier transform*. The Fourier transform with respect to the position variables is

defined by

$$\widehat{u}(k,t) := \int_{\mathbb{R}^d} u(x,t)e^{-ik\cdot x}dx\,.$$

Its inverse is given by

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^d} \widehat{u}(k,t)e^{ik\cdot x}dk\,.$$

The latter equation is certainly true for smooth u decaying sufficiently fast as $|x| \to \infty$. The usefulness of the Fourier transform for our purposes is a consequence of the identity

$$\widehat{\nabla u} = ik\widehat{u}\,,$$

implying $\widehat{\Delta u} = -|k|^2\widehat{u}$. Application of the Fourier transform to the linearized equation results in the ordinary differential equation

$$\partial_t\widehat{u} = (f'(p_0) - D|k|^2)\widehat{u}\,,$$

and, thus, in the general solution

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^d} \widehat{u_I}(k)e^{\lambda(k)t+ik\cdot x}dk\,,$$

where $\widehat{u_I}$ is the Fourier transform of $u(t=0)$ and the equation $\lambda(k) = f'(p_0) - D|k|^2$ is called the *dispersion relation* of the linearized equation. Obviously the condition for (asymptotic) stability is the same as in the case of a bounded domain. Actually, the dispersion relation also provides the discrete values $\lambda_j = f'(p_0) + D\mu_j$ for appropriate choices of the *wave vector* k.

References

[1] J.D. Murray, *Mathematical Biology, I: An Introduction*, 3rd ed., Springer, New York, 2002.

[2] J.D. Murray, *Mathematical Biology, II: Spatial Models and Biomedical Applications*, 3rd, ed., Springer, New York, 2003.