On a mathematical model for hot carrier injection in semiconductors

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Abstract

We study a collisionless transport model for electrons in a semiconductor, and we perform an asymptotic analysis for low temperatures or large applied biases. We derive analytic relations for the built-in potential and for the current which flows through the structure.

KEY WORDS : Vlasov-Poisson equation, Asymptotic expansions, Child-Langmuir law.
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1 Introduction

The operation of a semiconductor device heavily depends on the injection mechanism of carriers into the active regions through the junctions with the access zones. The built-in potential barrier sitting at such a junction reflects the low energy carriers, and only allows the energetic carriers to penetrate the active regions. For a unipolar $N^+ - N^- - N^+$ structure for instance, the built-in potential at equilibrium is given by the relation

$$V_{bi} = U_{th} \log \frac{N^+}{N^-} ,$$

where $U_{th} = k_B T/q$ is the thermal voltage and N^+, N^- are respectively the doping densities of the N^+ and N^- regions. However, away from equilibrium and, in particular, at high injection levels, no rigorously proved analytical formula for the built-in potential is known.

In the present paper, we shall consider a collisionless transport model, the Vlasov-Poisson equation, and we shall perform an asymptotic analysis for large applied biases compared with the thermal voltage. Such an analysis has already been performed in the vacuum diode case (see [5], [6] or [9]), and gives rise to the so-called "Child-Langmuir" regime. The application of the Child-Langmuir theory to semiconductors has been performed in [13], but some particular features of carrier injection in semiconductors had not been taken into account. In particular, in [13] the built-in potential is not considered. In the vacuum diode theory [6], the built-in potential tends to zero as the ratio of the thermal voltage to the applied bias tends to zero.

We shall propose a scaling which provides a finite built-in potential barrier at the source junction, and a potential drop at the drain junction. This is consistent with potential profiles obtained by numerical simulations [1]. In such a scaling, we account for the fact that the doping density in the N^+ region is very large. If the distribution function in the N^+ regions is assumed Maxwellian, the built-in potential barrier is (in the limit of small thermal voltages compared with the applied bias) given by

$$V_{bi} = U_{th} \log \left(\frac{9}{8\sqrt{\pi}} \frac{\sqrt{U_{th}}\ q L^2 N^+}{\varepsilon_s} \frac{1}{\phi_A^{3/2}} \right) , \tag{1.1}$$

where q is the elementary charge, L is the length of the N^- region, ε_s is the medium permittivity, and ϕ_A is the applied bias. A similar potential drop is obtained at the drain junction. A slight correction to formula (1.1) must be incorporated to account for the doping density of the N^- region :

$$V_{bi} = U_{th} \log \left(\frac{1}{\Gamma} \frac{\sqrt{U_{th}}\ q L^2 N^+}{\varepsilon_s} \frac{1}{\phi_A^{3/2}} \right) , \tag{1.2}$$

where Γ is a function of $n_D = qL^2 N^-/\varepsilon_s \phi_A$ implicitely given by the relation

$$\int_0^1 \frac{du}{\sqrt{2\Gamma\sqrt{u/\pi} - 2n_D u}} = 1 . \tag{1.3}$$

The relations (1.2) and (1.3) are valid for n_D not too large, namely for $n_D < \pi^2/2$.

Within the framework of the same asymptotics, a formula for the current which flows through the structure is obtained. When the N^- doping density is neglected, this current is simply the Child-Langmuir current:

$$J = \frac{4}{9}\varepsilon_s\sqrt{\frac{2q}{m}}\frac{\phi_A^{3/2}}{L^2}, \tag{1.4}$$

where m is the effective electron mass. When the N^- doping density is incorporated, the current is given by

$$J = \frac{\Gamma}{2\sqrt{\pi}}\varepsilon_s\sqrt{\frac{2q}{m}}\frac{\phi_A^{3/2}}{L^2} \tag{1.5}$$

where Γ is still given by (1.3).

Since our analysis relies on the stationary Vlasov-Poisson model, we shall refer to [8], [11], [12], for the existence theory and to [5], [6], for earlier mathematical sudies of the Child-Langmuir asymptotics for vacuum diodes. We also point out that the incorporation of collision terms has been investigated in [3], [4], and that the influence of a backgroung doping density has been analyzed in [2]. However, these latter studies did not produce any potential barrier in the asymptotic regime, and thus, their relevance for semiconductors is questionable. The present analysis also indicates that an algorithmic treatment of the injection boundary conditions in the spirit of [7] would be possible, and would perhaps improve or, at least, facilitate the numerical simulations of the semiconductor Boltzmann equation. This, together with comparisons between formulae (1.2) – (1.5) and (numerical) experiments, will be investigated in future work.

2 Scaling

We consider a one-dimensional unipolar semiconductor structure which consists of two highly doped N^+ regions on each side of a lowly doped N^- region. We solve for the electron distribution $F(X,V)$ in the N^- region, where $X \in [0,L], V \in I\!R$, and L is the length of the N^- region. We shall consider that $X = 0$ stands for the $N^+ - N^-$ junction on the source side, and $X = L$ for the junction at the drain side. Neglecting collisions, the equation for F is the one-dimensional stationary Vlasov equation:

$$V\frac{\partial F}{\partial X} + \frac{q}{m}\frac{d\Phi}{dX}\frac{\partial F}{\partial V} = 0, \qquad X \in [0,L], \quad V \in I\!R, \tag{2.1}$$

where q is the absolute value of the elementary charge, m is the electron effective mass in the medium, and $\Phi(X)$, $X \in [0,L]$ is the electrostatic potential. Φ solves the Poisson equation

$$\frac{d^2\Phi}{dX^2} = \frac{q}{\varepsilon_s}(N(X) - N_D(X)), \qquad X \in [0,L]\,, \tag{2.2}$$

where $N(X)$ is the electron density, given by

$$N(X) = \int_{-\infty}^{\infty} F(X,V)dV\,, \tag{2.3}$$

and $N_D(X)$ is the background doping density in the N^- region. In all the sequel, we shall assume that $N_D(X) = N_D$ is constant.

We assume that at both ends of the N^- region the incoming distribution function is known, given from the equilibrium distributions in the N^+ regions. Not to be too specific, we shall denote by $\bar{g}_0(|v|)$ and $\bar{g}_1(|v|)$, the normalized electron distribution functions in the N^+ regions on the source and drain sides respectively, satisfying the normalization condition

$$\int_0^{\infty} \bar{g}_0(v)dv = \int_0^{\infty} \bar{g}_1(v)dv = \frac{1}{2}\,.$$

Then, we prescribe

$$\begin{aligned} F(0,V) &= \frac{N_0^+}{V_{th}}\bar{g}_0\left(\frac{V}{V_{th}}\right), \qquad V > 0\,, \\ F(L,V) &= \frac{N_1^+}{V_{th}}\bar{g}_1\left(\frac{|V|}{V_{th}}\right), \qquad V < 0\,, \end{aligned} \tag{2.4}$$

where N_0^+ and N_1^+ are the doping densities of the N^+ regions on the source and drain sides respectively, and

$$V_{th}^2 = \frac{2k_BT}{m} \tag{2.5}$$

is the thermal velocity associated with the lattice temperature T, assumed to be uniform in the whole structure. By k_B we denote the Boltzmann constant. A fundamental example for the incoming distribution functions is given by two Maxwellian distributions

$$\bar{g}_0(v) = \bar{g}_1(v) = \frac{1}{\sqrt{\pi}}e^{-v^2}\,. \tag{2.6}$$

Finally, the potential of the source N^+ region is taken as reference potential, and the potential of the drain N^+ region is identified with the applied bias Φ_A. This leads to the following boundary conditions for the Poisson equation (2.2) :

$$\phi(0) = 0, \qquad \phi(L) = \phi_A > 0\,. \tag{2.7}$$

We now introduce a scaling, which is inspired from our previous studies of the so-called Child-Langmuir regime [6]. We consider the following scales for length $\bar{X}$, potential $\bar{\Phi}$, velocity $\bar{V}$, density $\bar{N}$, distribution function $\bar{F}$ and current $\bar{J}$:

$$\begin{array}{ll} \bar{X} = L, & \bar{\Phi} = \phi_A, \\ \bar{V} = \sqrt{2q\phi_A/m}, & \bar{N} = \varepsilon_s\phi_A/qL^2, \\ \bar{F} = \bar{N}/\bar{V}, & \bar{J} = -e\bar{N}\bar{V}. \end{array} \tag{2.8}$$

The velocity $\bar{V}$ is the one reached at the drain by a particle starting from the source with velocity 0, if only the external field is considered. The density $\bar{N}$ is the one needed over a distance L to produce a potential difference of the order of ϕ_A. Notice that our choice of potential unit is not the thermal voltage

$$U_{th} = \frac{k_B T}{q} = \frac{m}{2q} V_{th}^2 \,, \tag{2.9}$$

as commonly used in semiconductor studies [10]. This is because our main concern is high injection levels, where ϕ_A is large compared to U_{th}.

We now use lower-script letters for variables after scaling. The equations then become

$$v \frac{\partial f}{\partial x} + \frac{1}{2} \frac{d\varphi}{dx} \frac{\partial f}{\partial v} = 0 \,, \qquad x \in (0,1), \quad v \in I\!R, \tag{2.10}$$

$$\varphi'' = n - n_D \,, \qquad x \in (0,1), \tag{2.11}$$

$$n(x) = \int_{-\infty}^{\infty} f(x,v) dv \,, \qquad x \in (0,1), \tag{2.12}$$

with the scaled version $n_D = N_D/\bar{N}$ of the doping density in the N^- region.

The scaled boundary conditions (2.4) for the distribution function are

$$\begin{aligned} f(0,v) &= \frac{1}{\varepsilon\lambda^2} \bar{g}_0 \left(\frac{v}{\varepsilon} \right), \qquad v > 0 \,, \\ f(1,v) &= \frac{1}{\varepsilon\lambda^2} \left(\frac{N_1^+}{N_0^+} \right) \bar{g}_1 \left(\frac{|v|}{\varepsilon} \right), \qquad v < 0 \,, \end{aligned} \tag{2.13}$$

where we have introduced the dimensionless parameters

$$\varepsilon = \sqrt{\frac{U_{th}}{\phi_A}}, \qquad \lambda = \frac{\lambda_D}{L} = \frac{1}{L} \sqrt{\frac{\varepsilon_s \phi_A}{q N_0^+}} \,. \tag{2.14}$$

The high injection level assumption means $\varepsilon \ll 1$. Also in applications the Debye length

$$\lambda_D = \sqrt{\frac{\varepsilon_s \phi_A}{q N_0^+}}$$

of the source N^+ region will always be much smaller than the length of the N^- region. Therefore, we assume $\lambda^2 \ll 1$. The remaining dimensionless parameters n_D and N_1^+/N_0^+ are assumed to be $O(1)$. This means that the doping densities of the two N^+ regions are of the same order of magnitude. For simplicity, we introduce

$$g_0(v) = \bar{g}_0(v), \qquad g_1(v) = \frac{N_1^+}{N_0^+}\bar{g}_1(v),$$

so that (2.13) is written as

$$\begin{aligned} f(0,v) &= \frac{1}{\varepsilon\lambda^2} g_0\left(\frac{v}{\varepsilon}\right), \qquad v > 0\,, \\ f(1,v) &= \frac{1}{\varepsilon\lambda^2} g_1\left(\frac{|v|}{\varepsilon}\right), \qquad v < 0\,. \end{aligned} \tag{2.15}$$

Finally, the boundary conditions (2.7) for the potential turn into

$$\varphi(0) = 0\,, \qquad \varphi(1) = 1\,. \tag{2.16}$$

Let us now summarize the problem. We are interested in a situation where λ is connected to ε by some relation λ^ε which will be introduced later. Then, we are looking at the behaviour when $\varepsilon \to 0$ of the solution $(f^\varepsilon, \varphi^\varepsilon, n^\varepsilon)$, of the following problem:

$$v\frac{\partial f^\varepsilon}{\partial x} + \frac{1}{2}\frac{d\varphi^\varepsilon}{dx}\frac{\partial f^\varepsilon}{\partial v} = 0\,, \quad x \in (0,1)\,, \quad v \in I\!R\,, \tag{2.17}$$

$$(\varphi^\varepsilon)'' = n^\varepsilon - n_D\,, \quad x \in (0,1)\,, \tag{2.18}$$

$$n^\varepsilon(x) = \int_{-\infty}^{\infty} f^\varepsilon(x,v)dv\,, \quad x \in (0,1)\,, \tag{2.19}$$

$$\left\{ \begin{aligned} f^\varepsilon(0,v) &= \frac{1}{\varepsilon(\lambda^\varepsilon)^2} g_0\left(\frac{v}{\varepsilon}\right), \quad v > 0\,, \\ f^\varepsilon(1,v) &= \frac{1}{\varepsilon(\lambda^\varepsilon)^2} g_1\left(\frac{v}{\varepsilon}\right), \quad v < 0\,, \end{aligned} \right. \tag{2.20}$$

$$\varphi^\varepsilon(0) = 0\,, \qquad \varphi^\varepsilon(1) = 1\,. \tag{2.21}$$

In the previous studies on the Child-Langmuir asymptotics [3], [5], [6], we used $(\lambda^\varepsilon)^2 = \varepsilon$, $g_1 = 0$; and we proved that $\varphi^\varepsilon(x) \to \varphi^0(x)$ where φ^0 satisfies the boundary condition $\varphi^0(0) = 0$. This means that the potential barrier (if any) of $\varphi^\varepsilon(x)$ tends to zero as $\varepsilon \to 0$. In the present paper, we want $\varphi^0(0) = -\beta < 0$, which means that the potential barrier remains finite as $\varepsilon \to 0$. For this to happen, λ^ε must be connected with ε by a relation which specifically depends on the asymptotic behaviour of $g_0(v)$ and $g_1(v)$ when $v \to \infty$. We shall assume that

$$g_0(v) \sim A_0 e^{-v^2}\,, \quad g_1(v) \sim A_1 e^{-v^2}\,, \qquad \text{as } v \to \infty\,. \tag{2.22}$$

Other asymptotic behaviours could be analyzed as well, but the assumption (2.22) includes the practically relevant cases, e.g. the Maxwellian, with

$$A_0 = \frac{1}{\sqrt{\pi}}\,, \qquad A_1 = \frac{1}{\sqrt{\pi}}\frac{N_1^+}{N_0^+}. \tag{2.23}$$

In the case (2.22), λ^ε has to be given by

$$(\lambda^\varepsilon)^2 = \varepsilon e^{-\beta/\varepsilon^2}\,, \tag{2.24}$$

as we shall see later on. The aim of the paper is thus to study the behaviour of the solutions of (2.17)-(2.21) when (2.22) and (2.24) hold.

We conclude this section with a typical example for the values of the physical parameters. Let us consider a N^+-N-N^+ structure, with the following parameters :

$$L = 0.4\mu m, \qquad \phi_A = 1 \text{ Volt}\ ,$$

$$N_0^+ = N_1^+ = 10^{18} cm^{-3}, \qquad N_D = 2.10^{15} cm^{-3},$$

$$T = 77K.$$

Then, we have

$$\varepsilon \simeq 0.085, \qquad n_D \simeq 2.0, \qquad \lambda^2 = 0.004$$

and

$$\beta \simeq 0.02,$$

which means that a potential barrier of the order of 0.02 Volts should be expected. A more accurate value of the potential barrier β^ε will be deduced from the ansatz

$$\beta^\varepsilon = \beta - \gamma\varepsilon^2\,,$$

where in the case $n_D = 0$ we shall obtain $\gamma = \log \frac{8}{9A_0}$ and if $0 < n_D < \pi^2/2$, γ is determined from

$$\int_0^1 \frac{d\psi}{\sqrt{2A_0 e^\gamma \sqrt{\psi} - 2n_D\psi}} = 1\,.$$

In particular, for $n_D = 0$, and in the case of the Maxwellian (2.23), we obtain formula (1.1) for the built-in potential $V_{bi} = \beta\phi_A$.

3 Solution of the problem for finite ε

In [12], a general existence theory for the stationary Vlasov-Poisson system is given. However, in the one-dimensional case, explicit solutions can be constructed by the method of characteristics (see [6], [8]), facilitating the analysis of the asymptotic behaviour as $\varepsilon \to 0$.

We shall first treat the case $n_D = 0$. In this case, it can be proven [8] that φ^ε is a strictly convex function which has a unique minimum $\varphi_c^\varepsilon \leq 0$ at a point $x_c^\varepsilon \in [0,1)$. We shall consider only the case where $x_c^\varepsilon \in (0,1), \varphi_c^\varepsilon < 0$, which implies that $(\varphi^\varepsilon)'(x_c^\varepsilon) = 0$. Then, the phase space $[0,1] \times I\!R^3$ is separated into two regions by the critical characteristic

$$v = v_c(x) := \text{ sign } (x - x_c^\varepsilon)\sqrt{\varphi^\varepsilon(x) - \varphi_c^\varepsilon}. \tag{3.1}$$

We introduce the notation

$$\Omega_0 = \{(x,v) \in [0,1] \times I\!R, \ v > v_c(x)\},$$

$$\Omega_1 = \{(x,v) \in [0,1] \times I\!R, \ v < v_c(x)\},$$

and note that in Ω_0 the characteristics come from the left boundary, while in Ω_1 they come from the right. Therefore, we can write

$$f^\varepsilon(x,v) = \begin{cases} \dfrac{1}{(\lambda^\varepsilon)^2\varepsilon}\, g_0\left(\dfrac{\sqrt{v^2 - \varphi^\varepsilon(x)}}{\varepsilon}\right), & (x,v) \in \Omega_0\,, \\[2ex] \dfrac{1}{(\lambda^\varepsilon)^2\varepsilon}\, g_1\left(\dfrac{\sqrt{v^2 - \varphi^\varepsilon(x) + 1}}{\varepsilon}\right), & (x,v) \in \Omega_1\,. \end{cases} \tag{3.2}$$

Indeed, if $(x,v) \in \Omega_0$, the characteristic through (x,v) hits the left boundary at the point $(0,u)$ such that $v^2 - \varphi^\varepsilon(x) = \text{constant} = u^2 - \varphi^\varepsilon(0) = u^2$. On the other hand, if $(x,v) \in \Omega_1$, the characteristic through (x,v) hits the right boundary at the point $(0,-u)$, such that $v^2 - \varphi^\varepsilon(x) = u^2 - \varphi^\varepsilon(1) = u^2 - 1$. This, together with the boundary conditions (2.20) leads to the representation (3.2). For the details, we refer the reader to [6] or [8].

The density can be easily computed from (3.2). Its expression depends upon the position of x with respect to x_c^ε, because the critical characteristic (3.1) lies below the axes $v = 0$ for $x < x_c^\varepsilon$, and above it for $x > x_c^\varepsilon$. Therefore, we find :

For $x \leq x_c^\varepsilon$:

$$\begin{aligned} n^\varepsilon(x) \ &= \ \int_{\sqrt{\varphi^\varepsilon(x) - \varphi_c^\varepsilon}}^{\infty} \frac{1}{(\lambda^\varepsilon)^2\varepsilon}\ g_0\left(\frac{\sqrt{v^2 - \varphi^\varepsilon(x)}}{\varepsilon}\right) dv \\ &\quad + 2\int_0^{\sqrt{\varphi^\varepsilon(x) - \varphi_c^\varepsilon}} \frac{1}{(\lambda^\varepsilon)^2\varepsilon}\ g_0\left(\frac{\sqrt{v^2 - \varphi^\varepsilon(x)}}{\varepsilon}\right) dv \\ &\quad + \int_{\sqrt{\varphi^\varepsilon(x) - \varphi_c^\varepsilon}}^{\infty} \frac{1}{(\lambda^\varepsilon)^2\varepsilon}\ g_1\left(\frac{\sqrt{v^2 - \varphi^\varepsilon(x) + 1}}{\varepsilon}\right) dv \end{aligned} \tag{3.3}$$

For $x \geq x_c^\varepsilon$:

$$
\begin{aligned}
n^\varepsilon(x) \quad = \quad & \int_{\sqrt{\varphi^\varepsilon(x)-\varphi_c^\varepsilon}}^{\infty} \frac{1}{(\lambda^\varepsilon)^2\varepsilon} \; g_0\left(\frac{\sqrt{v^2-\varphi^\varepsilon(x)}}{\varepsilon}\right) dv \\
& + \int_{\sqrt{\varphi^\varepsilon(x)-\varphi_c^\varepsilon}}^{\infty} \frac{1}{(\lambda^\varepsilon)^2\varepsilon} \; g_1\left(\frac{\sqrt{v^2-\varphi^\varepsilon(x)+1}}{\varepsilon}\right) dv \\
& + 2\int_0^{\sqrt{\varphi^\varepsilon(x)-\varphi_c^\varepsilon}} \frac{1}{(\lambda^\varepsilon)^2\varepsilon} \; g_1\left(\frac{\sqrt{v^2-\varphi^\varepsilon(x)+1}}{\varepsilon}\right) dv
\end{aligned}
\tag{3.4}
$$

From (3.3) and (3.4) we see that, once φ_c^ε is known, $n^\varepsilon(x)$ depends on x only through $\varphi^\varepsilon(x)$. Therefore, we shall write:

$$
\text{For } x \leq x_c^\varepsilon : \quad n^\varepsilon(x) = n_\varepsilon^-(\varphi_c^\varepsilon, \varphi^\varepsilon(x)) \, .
$$

$$
\text{For } x \geq x_c^\varepsilon : \quad n^\varepsilon(x) = n_\varepsilon^+(\varphi_c^\varepsilon, \varphi^\varepsilon(x)) \, .
$$

By the changes of variables $u = \sqrt{v^2 - \varphi}$ or $u = \sqrt{v^2 - \varphi + 1}$ we can write

$$
\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 n_\varepsilon^+(\varphi_c, \varphi) \quad = \quad & \int_{\sqrt{-\varphi_c}}^{\infty} \; g_0\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi}} \\
& + \int_{\sqrt{1-\varphi_c}}^{\infty} \; g_1\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi-1}} \\
& + 2\int_{\sqrt{1-\varphi}}^{\sqrt{1-\varphi_c}} \; g_1\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi-1}} \, ,
\end{aligned}
\tag{3.5}
$$

for $\varphi_c \leq \varphi \leq 1$, and

$$
\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 n_\varepsilon^+(\varphi_c, \varphi) \quad = \quad & \int_{\sqrt{-\varphi_c}}^{\infty} \; g_0\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi}} \\
& + 2\int_{\sqrt{-\varphi}}^{\sqrt{-\varphi_c}} g_0\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi}} \\
& + \int_{\sqrt{1-\varphi_c}}^{\infty} \; g_1\left(\frac{u}{\varepsilon}\right) \frac{u du}{\sqrt{u^2+\varphi-1}} \, ,
\end{aligned}
\tag{3.6}
$$

for $\varphi_c \leq \varphi \leq 0$ (we recall that $\varphi_c < 0$ and that φ is convex).

It will be convenient to introduce the primitives of $n_\varepsilon^+(\varphi_c, \varphi)$ and $n_\varepsilon^-(\varphi_c, \varphi)$ with respect to φ, which vanish at $\varphi = \varphi_c$. These primitives will be denoted by $\mathcal{E}_\varepsilon^+(\varphi_c, \varphi)$, and $\mathcal{E}_\varepsilon^-(\varphi_c, \varphi)$, and are given by

$$
\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 \mathcal{E}_\varepsilon^+(\varphi_c, \varphi) \quad = \quad & 2\int_{\sqrt{-\varphi_c}}^{\infty} \; g_0\left(\frac{u}{\varepsilon}\right) \left(\sqrt{u^2+\varphi} - \sqrt{u^2+\varphi_c}\right) u \, du \\
& + 2\int_{\sqrt{1-\varphi_c}}^{\infty} \; g_1\left(\frac{u}{\varepsilon}\right) \left(\sqrt{u^2+\varphi-1} - \sqrt{u^2+\varphi_c-1}\right) u \, du \\
& + 4\int_{\sqrt{1-\varphi}}^{\sqrt{1-\varphi_c}} g_1\left(\frac{u}{\varepsilon}\right) \sqrt{u^2+\varphi-1} \; u \, du,
\end{aligned}
\tag{3.7}
$$

for $\varphi_c \le \varphi \le 1$, and

$$\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 \mathcal{E}_\varepsilon^-(\varphi_c,\varphi) \quad &= \quad 2\int_{\sqrt{-\varphi_c}}^{\infty} g_0\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi}-\sqrt{u^2+\varphi_c}\right)u\,du \\
&+4\int_{\sqrt{-\varphi}}^{\sqrt{-\varphi_c}} g_0\left(\frac{u}{\varepsilon}\right)\sqrt{u^2+\varphi}\;\,u\,du \\
&+2\int_{\sqrt{1-\varphi_c}}^{\infty} g_1\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi-1}-\sqrt{u^2+\varphi_c-1}\right)u\,du\,,
\end{aligned} \tag{3.8}$$

for $\varphi_c \le \varphi \le 0$.

We are now concerned with the Poisson equation

$$(\varphi^\varepsilon)'' = n^\varepsilon\,, \qquad x\in[0,1]\,, \tag{3.9}$$

$$\varphi^\varepsilon(0) = 0\,, \qquad \varphi^\varepsilon(1) = 1\,, \tag{3.10}$$

which we integrate separately on $[0,x_c^\varepsilon]$ and on $[x_c^\varepsilon,1]$. We multiply (3.9) by $(\varphi^\varepsilon)'$ and we integrate on $[x_c^\varepsilon,1]$. Since we have $(\varphi^\varepsilon)'(x_c^\varepsilon)=0$, this leads to

$$\frac{d}{dx}\left(\left(\frac{d\varphi^\varepsilon}{dx}\right)^2\right) \quad = \quad 2\;\, n_\varepsilon^+(\varphi_c^\varepsilon,\varphi^\varepsilon(x))\frac{d\varphi^\varepsilon}{dx} = 2\;\,\frac{d}{dx}\mathcal{E}_\varepsilon^+(\varphi_c^\varepsilon,\varphi^\varepsilon(x))$$

and

$$\left(\frac{d\varphi^\varepsilon}{dx}\right)^2(x) = 2\;\,\mathcal{E}_\varepsilon^+(\varphi_c^\varepsilon,\varphi^\varepsilon(x))\,, \quad x\in[x_c^\varepsilon,1]\,. \tag{3.11}$$

Since φ^ε is increasing on $[x_c^\varepsilon,1]$, we take the positive square root of (3.11) and get

$$\frac{(\varphi^\varepsilon(x))'}{\sqrt{2\mathcal{E}_\varepsilon^+(\varphi_c^\varepsilon,\varphi^\varepsilon(x))}} = 1\,, \quad x\in[x_c^\varepsilon,1]\,, \tag{3.12}$$

or, equivalently,

$$\int_{\varphi_c^\varepsilon}^{\varphi^\varepsilon(x)} \frac{d\psi}{\sqrt{2\mathcal{E}_\varepsilon^+(\varphi_c^\varepsilon,\psi)}} = x - x_c^\varepsilon\,, \quad x\in[x_c^\varepsilon,1]\,. \tag{3.13}$$

By the translation $\psi = \varphi_c^\varepsilon + \tilde\psi$, (3.13) can also be written

$$\int_0^{\varphi^\varepsilon(x)-\varphi_c^\varepsilon} \frac{d\psi}{\sqrt{2\mathcal{E}_\varepsilon^+(\varphi_c^\varepsilon,\varphi_c^\varepsilon+\psi)}} = x - x_c^\varepsilon\,, \quad x\in[x_c^\varepsilon,1]\,. \tag{3.14}$$

We define $G_\varepsilon^+(\varphi_c,\psi)$ and $H_\varepsilon^+(\varphi_c,\chi)$ according to

$$G_\varepsilon^+(\varphi_c, \psi) = 2\mathcal{E}_\varepsilon^+(\varphi_c, \varphi_c + \psi)\,, \quad 0 \le \psi \le 1 - \varphi_c\,, \tag{3.15}$$

$$H_\varepsilon^+(\varphi_c, \chi) = \int_0^\chi \frac{d\psi}{\sqrt{G_\varepsilon^+(\varphi_c, \psi)}}\,, \quad 0 \le \chi \le 1 - \varphi_c\,, \tag{3.16}$$

so that (3.14) is written

$$H_\varepsilon^+(\varphi_c^\varepsilon, \varphi^\varepsilon(x) - \varphi_c^\varepsilon) = x - x_c^\varepsilon\,, \quad x \in [x_c^\varepsilon, 1]\,. \tag{3.17}$$

The same computation on $[0, x_c^\varepsilon]$ leads to

$$H_\varepsilon^-(\varphi_c^\varepsilon, \varphi^\varepsilon(x) - \varphi_c^\varepsilon) = x_c^\varepsilon - x\,, \quad x \in [0, x_c^\varepsilon]\,, \tag{3.18}$$

with

$$H_\varepsilon^-(\varphi_c, \chi) = \int_0^\chi \frac{d\psi}{G_\varepsilon^-(\varphi_c, \psi)}\,, \quad 0 \le \chi \le -\varphi_c\,, \tag{3.19}$$

$$G_\varepsilon^-(\varphi_c, \psi) = 2\mathcal{E}_\varepsilon^-(\varphi_c, \varphi_c + \psi)\,, \quad 0 \le \psi \le -\varphi_c\,. \tag{3.20}$$

Now, the existence of $\varphi^\varepsilon(x)$ reduces to that of x_c^ε and φ_c^ε, which are such that $\varphi^\varepsilon(x)$ matches the boundary conditions (3.10). This leads to the equations

$$H_\varepsilon^+(\varphi_c^\varepsilon, 1 - \varphi_c^\varepsilon) = 1 - x_c^\varepsilon\,, \tag{3.21}$$

$$H_\varepsilon^-(\varphi_c^\varepsilon, -\varphi_c^\varepsilon) = x_c^\varepsilon\,. \tag{3.22}$$

The position of the minimum x_c^ε can be eliminated from this system, and we find the following equation for φ_c^ε:

$$F_\varepsilon(\varphi_c^\varepsilon) = H_\varepsilon^+(\varphi_c^\varepsilon, 1 - \varphi_c^\varepsilon) + H_\varepsilon^-(\varphi_c^\varepsilon, -\varphi_c^\varepsilon)) = 1\,. \tag{3.23}$$

We now have the following

Lemma 3.1 *Let g_0 and g_1 be non increasing. Then F_ε is a decreasing function of φ_c.*

Proof: The proof is analogous to that of [6], Lemma 5.4: Obviously, H_ε^+ and H_ε^- are increasing functions of χ. That G_ε^+ and G_ε^- are non decreasing functions of φ_c follows from taking the derivatives with respect to φ_c, integrating by parts and showing that the resulting expression is non negative. The details are left to the reader. ■

Remark: The fact that the equation (3.12) can be integrated according to (3.13), and that the integral in (3.13) is defined can also be derived from [6].

We shall not go further into the proof of the existence of φ_c^ε. This will follow from the subsequent asymptotic analysis, which will give us the existence of φ_c^ε for small ε, and the asymptotic behaviour of φ_c^ε when $\varepsilon \to 0$.

4 Asymptotic behaviour for $\varepsilon \to 0$

We recall the expressions of $G_\varepsilon^+(\varphi_c, \psi)$ and $G_\varepsilon^-(\varphi_c, \psi)$ from (3.15), (3.20) and (3.7), (3.8)

$$\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 G_\varepsilon^+(\varphi_c, \psi) &= 4\int_{\sqrt{-\varphi_c}}^\infty g_0\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi_c+\psi} - \sqrt{u^2+\varphi_c}\right) u\,du \\
&+ 4\int_{\sqrt{1-\varphi_c}}^\infty g_1\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi_c-1+\psi} - \sqrt{u^2+\varphi_c-1}\right) u\,du \\
&+ 8\int_{\sqrt{1-\varphi_c-\psi}}^{\sqrt{1-\varphi_c}} g_1\left(\frac{u}{\varepsilon}\right)\sqrt{u^2+\varphi_c-1+\psi}\; u\,du\,, \qquad (4.1)
\end{aligned}$$

for $0 \le \psi \le 1-\varphi_c$, and

$$\begin{aligned}
\varepsilon(\lambda^\varepsilon)^2 G_\varepsilon^-(\varphi_c, \psi) &= 4\int_{\sqrt{-\varphi_c}}^\infty g_0\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi_c+\psi} - \sqrt{u^2+\varphi_c}\right) u\,du \\
&+ 8\int_{\sqrt{-\varphi_c-\psi}}^{\sqrt{-\varphi_c}} g_0\left(\frac{u}{\varepsilon}\right)\sqrt{u^2+\varphi_c+\psi}\; u\,du \qquad (4.2) \\
&+ 4\int_{\sqrt{1-\varphi_c}}^\infty g_1\left(\frac{u}{\varepsilon}\right)\left(\sqrt{u^2+\varphi_c-1+\psi} - \sqrt{u^2+\varphi_c-1}\right) u\,du\,,
\end{aligned}$$

for $0 \le \psi \le -\varphi_c$.

Now we make the following changes of variables: $v^2 = (u^2+\varphi_c)/\varepsilon^2$ in the first integrals of (4.1) and (4.2), $v^2 = (u^2+\varphi_c-1)/\varepsilon^2$ in the second integral of (4.1) and in the third one of (4.2), $v^2 = (u^2+\varphi_c-1+\psi)/\varepsilon^2$ in the third integral of (4.1) and $v^2 = (u^2+\varphi_c+\psi)/\varepsilon^2$ in the second integral of (4.2). Then we get

$$\begin{aligned}
(\lambda^\varepsilon)^2 G_\varepsilon^+(\varphi_c, \psi) &= 4\varepsilon\int_0^\infty g_0\left(\sqrt{v^2-\frac{\varphi_c}{\varepsilon^2}}\right)\left(\sqrt{\varepsilon^2 v^2+\psi} - \varepsilon v\right) v\,dv \\
&+4\varepsilon\int_0^\infty g_1\left(\sqrt{v^2-\frac{\varphi_c-1}{\varepsilon^2}}\right)\left(\sqrt{\varepsilon^2 v^2+\psi} - \varepsilon v\right) v\,dv \\
&+8\varepsilon^2\int_0^{\sqrt{\psi}/\varepsilon} g_1\left(\sqrt{v^2-\frac{\varphi_c-1+\psi}{\varepsilon^2}}\right) v^2\,dv\,, \qquad (4.3)
\end{aligned}$$

for $0 \le \psi \le 1-\varphi_c$, and

$$\begin{aligned}
(\lambda^\varepsilon)^2 G_\varepsilon^-(\varphi_c, \psi) &= 4\varepsilon\int_0^\infty g_0\left(\sqrt{v^2-\frac{\varphi_c}{\varepsilon^2}}\right)\left(\sqrt{\varepsilon^2 v^2+\psi} - \varepsilon v\right) v\,dv \\
&+8\varepsilon^2\int_0^{\sqrt{\psi}/\varepsilon} g_0\left(\sqrt{v^2-\frac{\varphi_c+\psi}{\varepsilon^2}}\right) v^2\,dv \\
&+4\varepsilon\int_0^\infty g_1\left(\sqrt{v^2-\frac{\varphi_c-1}{\varepsilon^2}}\right)\left(\sqrt{\varepsilon^2 v^2+\psi} - \varepsilon v\right) v\,dv\,, \qquad (4.4)
\end{aligned}$$

for $0 \le \psi \le -\varphi_c$.

From the expressions (4.3) and (4.4), it is possible to obtain the asymptotic behaviour of $H_\varepsilon^+(\varphi_c, \chi)$ and $H_\varepsilon^-(\varphi_c, \chi)$ for $\varepsilon \to 0$. More precisely, we have

Lemma 4.1 *We assume that (2.22) holds. Let $\beta > 0$, $\gamma \in I\!R$. Then, as $\varepsilon \to 0$, we have*

$$\lim_{\varepsilon \to 0} H_\varepsilon^+(-\beta + \gamma\varepsilon^2, \chi) = \frac{4}{3\sqrt{2A_0}} e^{-\gamma/2} Min(\chi^{3/4}, 1) \tag{4.5}$$

for $0 < \chi < 1 + \beta$, and

$$\lim_{\varepsilon \to 0} H_\varepsilon^-(-\beta + \gamma\varepsilon^2, \chi) = 0 \tag{4.6}$$

for $0 < \chi < \beta$.

Proof : We shall denote by $I_\varepsilon^i(\varphi_c, \psi)$, $i = 1, 2, 3$, the three integrals appearing on the right hand side of (4.3), and by $I_\varepsilon^4(\varphi_c, \psi)$ the second one on the right hand side of (4.4). Then,

$$I_\varepsilon^1(-\beta + \gamma\varepsilon^2, \psi) \quad = \quad 4\varepsilon \int_0^\infty g_0 \left(\sqrt{v^2 + \frac{\beta}{\varepsilon^2} - \gamma} \right) \left(\sqrt{\varepsilon^2 v^2 + \psi} - \varepsilon v \right) v \, dv \,,$$

and, with the asymptotic behaviour (2.22), it follows that, as $\varepsilon \to 0$,

$$I_\varepsilon^1(-\beta + \gamma\varepsilon^2, \psi) \sim (4A_0 e^\gamma)\varepsilon e^{-\beta/\varepsilon^2} \int_0^\infty e^{-v^2} \left(\sqrt{\varepsilon^2 v^2 + \psi} - \varepsilon v \right) v \, dv \,.$$

Then

$$\int_0^\infty e^{-v^2} \left(\sqrt{\varepsilon^2 v^2 + \psi} - \varepsilon v \right) v \, dv \longrightarrow \frac{1}{2}\sqrt{\psi} \,,$$

so that

$$I_\varepsilon^1(-\beta + \gamma\varepsilon^2, \psi) \sim (\lambda^\varepsilon)^2 \; 2A_0 e^\gamma \sqrt{\psi} \,, \quad \varepsilon \to 0 \,. \tag{4.7}$$

A similar analysis shows that

$$\begin{aligned} I_\varepsilon^2(-\beta + \gamma\varepsilon^2, \psi) \quad &\sim \quad (\lambda^\varepsilon)^2 e^{-1/\varepsilon^2} \; 2A_1 e^\gamma \sqrt{\psi} \\ &\ll I_\varepsilon^1(-\beta + \gamma\varepsilon^2, \psi) \,, \quad \varepsilon \to 0 \,. \end{aligned} \tag{4.8}$$

For $0 < \psi < \beta + 1$ we have, with (2.22),

$$\begin{aligned} I_\varepsilon^3(-\beta + \gamma\varepsilon^2, \psi) \quad &\sim \quad 8\varepsilon^2 A_1 e^\gamma e^{-(\beta+1-\psi)/\varepsilon^2} \int_0^\infty e^{-v^2} v^2 dv \\ &= \quad 2A_1 \sqrt{\pi} e^\gamma \varepsilon^2 e^{-(\beta+1-\psi)/\varepsilon^2} \,, \quad \varepsilon \to 0 \,. \end{aligned} \tag{4.9}$$

Similary, for $0 < \psi < \beta$, we have

$$I^4_\varepsilon(-\beta+\gamma\varepsilon^2,\psi) \sim 2A_0\sqrt{\pi}e^\gamma\varepsilon^2 e^{-(\beta-\psi)/\varepsilon^2}\,, \quad \varepsilon\to 0\,. \tag{4.10}$$

For $0<\psi<1$ we have $\varepsilon^2 e^{-(\beta+1-\psi)/\varepsilon^2} \ll \varepsilon e^{-\beta/\varepsilon^2} = (\lambda^\varepsilon)^2$, such that

$$\lim_{\varepsilon\to 0} G^+_\varepsilon(-\beta+\gamma\varepsilon^2,\psi) = 2A_0 e^\gamma\sqrt{\psi}\,, \quad 0<\psi<1\,, \tag{4.11}$$

holds. On the other hand, when $1<\psi<1+\beta$, we have $\varepsilon^2 e^{-(\beta+1-\psi)/\varepsilon^2} \gg (\lambda^\varepsilon)^2$, so that

$$G^+_\varepsilon(-\beta+\gamma\varepsilon^2,\psi) \sim (2A_1\sqrt{\pi}e^\gamma)\varepsilon\, e^{(\psi-1)/\varepsilon^2}\,, \quad 1<\psi<1+\beta\,. \tag{4.12}$$

We now prove (4.5) for χ such that $0\le\chi\le 1$. For that purpose, it is sufficient to prove that the function $Y(\varepsilon,\psi) = G^+_\varepsilon(-\beta+\gamma\varepsilon^2,\psi)^{-1/2}$ is bounded by an integrable function of ψ on $[0,1]$, independently of ε. Then, Lebesgue's theorem will give the result. But

$$Y(\varepsilon,\psi) \le \lambda^\varepsilon I^1_\varepsilon(-\beta+\gamma\varepsilon^2,\psi)^{-1/2}\,. \tag{4.13}$$

On the other hand, for $v<1/4\varepsilon$ and $\psi<1$ we have

$$\sqrt{\varepsilon^2v^2+\psi} - \varepsilon v \ge \frac{\psi}{2}\,. \tag{4.14}$$

From (2.22), for ε small enough, we obtain

$$g_0\left(\sqrt{v^2+\frac{\beta}{\varepsilon^2}-\gamma}\right) \ge \frac{A_0}{2}e^\gamma e^{-\beta/\varepsilon^2}\,. \tag{4.15}$$

Thus (4.14) and (4.15) imply that, for ε small,

$$I^1_\varepsilon(-\beta+\gamma\varepsilon^2,\psi) \;\ge\; (A_0e^\gamma)(\lambda^\varepsilon)^2\left(\int_0^{1/4\varepsilon} ve^{-v^2}\,dv\right)\psi \ge C(\lambda^\varepsilon)^2\psi$$

with $C>0$ independent of ε. Therefore, with (4.13) :

$$Y(\varepsilon,\psi) \le \frac{1}{\sqrt{C\psi}} \in L^1(0,1),$$

proving the result for $0\le\chi\le 1$.

Now we prove (4.5) for χ such that $1\le\chi\le 1+\beta$. We have, still with $Y(\varepsilon,\psi) = G^+_\varepsilon(-\beta+\gamma\varepsilon^2,\psi)^{-1/2}$,

$$H^+_\varepsilon(-\beta+\gamma\varepsilon^2,\chi) = H^+_\varepsilon(-\beta+\gamma\varepsilon^2,1) + \int_1^\chi Y(\varepsilon,\psi)\;d\psi\,,$$

and, from the previous result,

$$\lim_{\varepsilon\to 0} H_\varepsilon^+(-\beta+\gamma\varepsilon^2,1) = \frac{4}{3\sqrt{2A_0}}e^{2\gamma}\,.$$

From (4.12) it follows that $Y(\varepsilon,\psi)\to 0$ as $\varepsilon\to 0$ for $1<\psi<1+\beta$. On the other hand, the same method as for $\psi<1$ can be applied and shows that $Y(\varepsilon,\chi)$ is bounded by an integrable function of ψ, independently of ε. Thus, by Lebesgue's theorem, we have

$$\lim_{\varepsilon\to 0}\int_1^\chi Y(\varepsilon,\psi)\,d\psi = 0\,.$$

This ends the proof of (4.5).

We now turn to (4.6). For $0<\psi<\beta$ we have $\varepsilon^2 e^{-(\beta-\psi)/\varepsilon^2} \gg \varepsilon e^{-\beta/\varepsilon^2} = (\lambda^\varepsilon)^2$. It follows that

$$G_\varepsilon^-(-\beta+\gamma\varepsilon^2,\psi) \sim 2A_0\sqrt{\pi}e^\gamma\varepsilon e^{\psi/\varepsilon^2}\,,\quad \text{as } \varepsilon\to 0\,.$$

Thus, $Y(\varepsilon,\psi) = G_\varepsilon^-(-\beta+\gamma\varepsilon^2,\psi)^{-1/2}\to 0$ as $\varepsilon\to 0$ for $0<\psi<\beta$, and the same reasoning applies again to show that $Y(\varepsilon,\psi)$ is bounded by an integrable function of ψ independently of ε. This proves (4.6). ■

Now we turn to the existence of a solution φ_c^ε of equation (3.23) :

Proposition 4.2 *: Let g_0 and g_1 be nonincreasing functions such that (2.22) holds. Then, for ε small enough, there exists a unique solution φ_c^ε of equation (3.23). Moreover, we have*

$$\varphi_c^\varepsilon = -\beta+\gamma\varepsilon^2+o(\varepsilon^2)\quad \textit{as } \varepsilon\to 0\,, \tag{4.16}$$

where

$$\gamma = \log\frac{8}{9A_0}\,. \tag{4.17}$$

Corollary 4.3 *: For ε small enough, there exists a unique solution $(f^\varepsilon,\varphi^\varepsilon,n^\varepsilon)$ of problem (2.17) – (2.21) with $n_D=0$.*

Proof of corollary 4.3: The existence of a solution follows from the existence of φ_c^ε. The uniqueness follows from the analysis of [8]. ■

Proof of proposition 4.2: We know that F_ε is a decreasing continuous function of φ_c. Let γ be given by (4.17), and γ_1,γ_2, such that $\gamma_1<\gamma<\gamma_2$. We define

$$\varphi_i^\varepsilon = -\beta+\gamma_i\varepsilon^2\,,\quad i=1,2.$$

From (4.5) and (4.6) we have

$$\lim_{\varepsilon\to 0} F_\varepsilon(\varphi_i^\varepsilon) = \frac{4}{3\sqrt{2A_0}}e^{-\gamma_i/2}\,.$$

Then for ε small, we have

$$F_\varepsilon(\varphi_1^\varepsilon) > 1 > F_\varepsilon(\varphi_2^\varepsilon).$$

Thus, there exists a unique φ_c^ε such that $F_\varepsilon(\varphi_c^\varepsilon) = 1$, and we have $\varphi_1^\varepsilon < \varphi_c^\varepsilon < \varphi_2^\varepsilon$. To prove (4.16), we define $\delta(\varepsilon)$ by

$$\varphi_c^\varepsilon = -\beta + \gamma\varepsilon^2 + \delta(\varepsilon)\varepsilon^2 \,. \tag{4.18}$$

Let us suppose that $\delta(\varepsilon)$ does not tend to 0. Then, there exists a constant $\alpha > 0$ and a subsequence $\varepsilon \to 0$, such that $|\delta(\varepsilon)| \geq \alpha > 0$. Let us suppose first that $\delta(\varepsilon) \geq \alpha > 0$. Then $F(\varphi_c^\varepsilon) \leq F(-\beta + (\gamma + \alpha)\varepsilon^2) \sim e^{-\alpha/2} < 1$, which contradicts that $F(\varphi_c^\varepsilon) = 1$. If, on the other hand, we have $\delta(\varepsilon) \leq -\alpha < 0$, then $F(\varphi_c^\varepsilon) \geq F(-\beta + (\gamma - \alpha)\varepsilon^2) \sim e^{\alpha/2} > 1$, which again leads to a contradiction. Therefore, $\delta(\varepsilon) \to 0$ and the proposition is proved. ■

We are now able to give the asymptotic behaviour of the potential:

Proposition 4.4 *We have*

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(x) = \varphi^0(x) = -\beta + x^{4/3} \,, \quad x \in (0,1) \,. \tag{4.19}$$

Remark: We have $\varphi^0(0) = -\beta$, $\varphi^0(1) = 1 - \beta$. In particular, $\varphi^\varepsilon(0) \not\to \varphi^0(0)$ and $\varphi^\varepsilon(1) \not\to \varphi^0(1)$. There appear a potential barrier at $x = 0$ and a potential drop at $x = 1$.

Proof of proposition 4.4: From (4.6), it is easy to show that

$$x_c^\varepsilon = H_\varepsilon^-(\varphi_c^\varepsilon, -\varphi_c^\varepsilon) \to 0 \,, \quad \text{as } \varepsilon \to 0 \,.$$

Thus, for $x > 0$ and ε small, we have $x > x_x^\varepsilon$ and $\varphi^\varepsilon(x)$ is given by

$$H_\varepsilon^+(\varphi_c^\varepsilon, \varphi^\varepsilon(x) - \varphi_c^\varepsilon) = x - x_c^\varepsilon \,. \tag{4.20}$$

Furthermore, since $x - x_c^\varepsilon$ is bounded away from 0 as $\varepsilon \to 0$, so is $\varphi^\varepsilon(x) - \varphi_c^\varepsilon$. Similarly, $\varphi^\varepsilon(x) - \varphi_c^\varepsilon < 1$ since $H_\varepsilon^+(\varphi_c^\varepsilon, 1) \to 1 > x - x_c^\varepsilon$ for ε small. It is easy to show that the limit (4.5) is uniform with respect to χ in any compact subinterval of $[0, \infty)$. Therefore, taking the equivalents of both sides of (4.20), we find

$$(\varphi^\varepsilon(x) - \varphi_c^\varepsilon)^{3/4} \sim x \quad \text{as } \varepsilon \to 0 \,, \qquad x \in (0,1) \,,$$

which gives (4.19). ■

We finally investigate the limit of the distribution function:

Proposition 4.5 *For any $x \in (0,1)$ we have*

$$f^\varepsilon(x,v) \to \frac{4}{9} \frac{1}{\sqrt{\varphi^0(x) + \beta}} \; \delta\left(v - \sqrt{\varphi^0(x) + \beta}\right) \qquad \textit{as } \varepsilon \to 0 \,, \tag{4.21}$$

in $\mathcal{M}_b(\mathbb{R})$, where δ is the delta function and $\mathcal{M}_b(\mathbb{R})$ is the space of bounded measures in $\mathbb{R}$.

Proof : Let $\theta(v)$ belong to the space $C^0_\infty(I\!R)$ of continuous functions on $I\!R$ which vanish at infinity. For $x \in (0,1)$ we define

$$I^\varepsilon_\theta(x) = \int_{-\infty}^{\infty} f^\varepsilon(x,v)\theta(v)dv. \tag{4.22}$$

Since $x > x^\varepsilon_c$ for small ε, we find $I^\varepsilon_\theta(x)$ by the same formula as (3.4) for $n^\varepsilon(x)$, but with dv replaced by $\theta(v)dv$. Then, performing the same changes of variables as those used to obtain the formula (4.3) for G^+_ε, we finally get

$$I^\varepsilon_\theta(x) = J^\varepsilon_\theta(\varphi^\varepsilon_c, \varphi^\varepsilon(x) - \varphi^\varepsilon_c) \tag{4.23}$$

with

$$\begin{aligned}(\lambda^\varepsilon)^2 J^\varepsilon_\theta(\varphi_c, \chi) \quad = \quad & \varepsilon \int_0^\infty g_0\left(\sqrt{v^2 - \frac{\varphi_c}{\varepsilon^2}}\right) \frac{\theta(\sqrt{\varepsilon^2 v^2 + \chi})}{\sqrt{\varepsilon^2 v^2 + \chi}}\, v dv \\ & + \varepsilon \int_0^\infty g_1\left(\sqrt{v^2 - \frac{\varphi_c - 1}{\varepsilon^2}}\right) \frac{\theta(\sqrt{\varepsilon^2 v^2 + \chi})}{\sqrt{\varepsilon^2 v^2 + \chi}}\, v dv \\ & + 2 \int_0^{\sqrt{\chi/\varepsilon}} g_1\left(\sqrt{v^2 - \frac{\varphi_c - 1 + \chi}{\varepsilon^2}}\right) \theta(\varepsilon v)\, dv\,. \end{aligned} \tag{4.24}$$

It is easy to see that

$$\lim_{\varepsilon \to 0} J^\varepsilon_\theta(\varphi^\varepsilon_c, \chi) = \frac{1}{2} A_0 e^\gamma \frac{\theta(\sqrt{\chi})}{\sqrt{\chi}} \tag{4.25}$$

holds uniformly for χ in compact subsets of $(0,1)$. Since for $x \in (0,1)$ and ε small, $\varphi^\varepsilon(x) - \varphi^\varepsilon_c$ lies in a compact subset of $(0,1)$, we have

$$\lim_{\varepsilon \to 0} I^\varepsilon_\theta(x) = \frac{1}{2} A_0 e^\gamma \frac{\theta(\sqrt{\varphi^0(x) + \beta})}{\sqrt{\varphi^0(x) + \beta}}\,. \tag{4.26}$$

Now, from (4.17),

$$\frac{1}{2} A_0 e^\gamma = \frac{4}{9} \tag{4.27}$$

and (4.21) follows from (4.26) and (4.27). ∎

Remark : Since $\varphi^0(x) + \beta = x^{4/3}$, the limiting distribution function, according to (4.21), corresponds to a monokinetic beam emitted from the source contact with velocity zero, and reaching the drain contact following the characterisitic $v(x) = x^{2/3}$. It has the same features as in the usual Child-Langmuir regime [6]. In particular, no particle coming from the drain contact is entering the domain. It should not be deduced that the influence of the drain boundary data g_1 is negligible. Indeed, the potential drop, from $1 - \beta$ to 1 is produced by g_1, as shown in Lemma 4.1 (formula (4.12).

From (4.21), it follows that the current j^0 which flows through the structure is

$$j^0 = \frac{4}{9},$$

(take $\theta(v) = v$ in (4.26)). It is the usual Child-Langmuir current [6]. Thus, the limiting density $n^0(x)$ is (take $\theta(v) = 1$ in (4.26))

$$n^0(x) = \frac{j^0}{\sqrt{\varphi^0(x) + \beta}}.$$

Now $\varphi^0(x) = x^{4/3} - \beta$ satisfies

$$(\varphi^0)'' = (\varphi^0 + \beta)'' = \frac{4}{9}x^{-2/3} = j^0/\sqrt{\varphi^0(x) + \beta} = n^0(x).$$

Thus, φ^0 satisfies the Poisson equation with boundary data $\varphi^0(0) = -\beta, \varphi^0(1) = 1 - \beta$.

5 Extension to $n_D \neq 0$

In this section, we investigate the case of a nonvanishing doping density n_D in the N^- region. We shall proceed in the same way as in the case $n_D = 0$, by assuming that the potential φ^ε has a unique minimum $\varphi_c^\varepsilon < 0$ at some point $x_c^\varepsilon \in (0, 1)$, and that it is decreasing on $[0, x_c^\varepsilon]$, and increasing on $[x_c^\varepsilon, 1]$. Then we shall show that, if n_D is not too large, such a solution exists for small ε. The method will follow very closely that of parts 3 and 4, and the proofs will only be sketched. The details are left to the reader.

Under the above mentioned assumptions on the potential φ^ε, the phase portrait of the characteristics is the same, and in particular formulae (3.2) up to (3.8) also hold in the present case. The Poisson equation now reads

$$(\varphi^\varepsilon)'' = n^\varepsilon - n_D, \quad x \in [0, 1], \tag{5.1}$$

$$\varphi^\varepsilon(0) = 0, \qquad \varphi^\varepsilon(1) = 1. \tag{5.2}$$

It is readily seen that its integration now gives rise to the same formulae

$$H_\varepsilon^+(\varphi_c^\varepsilon, \varphi^\varepsilon(x) - \varphi_c^\varepsilon) = x - x_c^\varepsilon, \quad x \in [x_c^\varepsilon, 1], \tag{5.3}$$

$$H_\varepsilon^-(\varphi_c^\varepsilon, \varphi^\varepsilon(x) - \varphi_c^\varepsilon) = x_c^\varepsilon - x, \quad x \in [0, x_c^\varepsilon], \tag{5.4}$$

but with

$$H_\varepsilon^\pm(\varphi_c, \chi) = \int_0^\chi \frac{d\psi}{\sqrt{G_\varepsilon^\pm(\varphi_c, \psi) - 2n_D\psi}}, \tag{5.5}$$

where $G_\varepsilon^\pm$ are given by (4.1), (4.2) or (4.3), (4.4). Of course, $H_\varepsilon^\pm$ are only defined for χ such that $G_\varepsilon^\pm(\varphi_c, \psi) - 2n_D\psi \geq 0, \ \forall \psi \in [0, \chi]$. We now prove

Lemma 5.1 *: We assume that (2.22) holds. Let $\beta > 0$, $\gamma > \log\left(\frac{n_D}{A_0}\right)$ hold. Then*

$$\lim_{\varepsilon\to 0} H_\varepsilon^+(-\beta+\gamma\varepsilon^2,\chi) = \int_0^{\min(\chi,1)} \frac{d\psi}{\sqrt{2A_0 e^\gamma\sqrt{\psi} - 2n_D\psi}}\,, \qquad \textit{for } 0<\chi<1+\beta \quad (5.6)$$

and

$$\lim_{\varepsilon\to 0} H_\varepsilon^-(-\beta+\gamma\varepsilon^2,\chi) = 0\,, \qquad \textit{for } 0<\chi<\beta\,. \tag{5.7}$$

Proof: The proof is slightly more involved than that of Lemma 4.1, because we must control the term $2n_D\psi$ in the denominator of (5.5). We prove (5.6) for $0\le\chi\le 1$: From (4.11), we obtain for $0\le\psi\le 1$

$$\lim_{\varepsilon\to 0} G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi = 2A_0e^\gamma\sqrt{\psi} - 2n_D\psi\,.$$

But for $\gamma > \log\frac{n_D}{A_0}$ and $0<\psi<1$, we have $2A_0e^\gamma\sqrt{\psi}-2n_D\psi > 0$. Then (5.6) for $0\le\chi\le 1$ will follow from Lebesgue's convergence theorem if we prove that

$$Y(\varepsilon,\psi) = (G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi)^{-1/2}$$

is bounded by an integrable function of ψ, independently of ε. We shall use

$$G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) \ge \frac{1}{(\lambda^\varepsilon)^2} I_\varepsilon^1(-\beta+\gamma\varepsilon^2,\psi)\,.$$

Let $\delta>0$ be arbitrary. For ε small enough, we have

$$g_0\left(\sqrt{v^2+\frac{\beta}{\varepsilon^2}-\gamma}\right) \ge (1-\delta)A_0e^\gamma e^{-\beta/\varepsilon^2}e^{-v^2}\,.$$

On the other hand,

$$\sqrt{\varepsilon^2v^2+\psi} - \varepsilon v \ge (1-\delta)\psi$$

holds for $0\le\psi\le 1$ and $v<\delta/2\varepsilon$. Thus, again for ε small

$$\begin{aligned} I_\varepsilon^1(-\beta+\gamma\varepsilon^2,\psi) &\ge (\lambda^\varepsilon)^2(1-\delta)^2\;2A_0e^\gamma\psi\left(\frac{1}{2}\int_0^{\delta/2\varepsilon} ve^{-v^2}dv\right)\\ &\ge (\lambda^\varepsilon)^2(1-\delta)^3\;2A_0e^\gamma\psi\,. \end{aligned}$$

Now, let $\gamma>\log\frac{n_D}{A_0}$. Then, there exists $\delta>0$ such that

$$(1-\delta)^3A_0e^\gamma > (1+\delta)n_D\,,$$

and, thus,

$$\begin{aligned} G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi &\ge (\lambda^\varepsilon)^{-2}I_\varepsilon^1(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi\\ &\ge 2[(1-\delta)^3A_0e^\gamma - n_D]\psi\\ &\ge 2\delta n_D\psi \ge C\psi\,,\quad C>0\,. \end{aligned}$$

Therefore,

$$Y(\varepsilon,\psi) \le \frac{1}{\sqrt{C\psi}} \in L^1(0,1)\,, \quad C>0 \tag{5.8}$$

holds, proving the result for $0 \le \chi \le 1$.

Next, we prove (5.6) for $1 \le \chi \le 1+\beta$. From (4.12), we have for $1 < \psi < 1+\beta$ and $\varepsilon \to 0$:

$$Y(\varepsilon,\psi) = (G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi)^{-1/2} \to 0\,.$$

Looking carefully at the previous proof, we can show that the result (5.8) is still true for $1 \le \psi \le 1+\zeta$ with ζ small enough (depending on γ but not on ε). Then, for $\psi \ge 1+\zeta$, according to (4.9) there exists $C>0$ such that

$$G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) \;\ge\; \lambda_\varepsilon^{-2} I_\varepsilon^3(-\beta+\gamma\varepsilon^2,\psi) \ge C\varepsilon e^{\zeta/\varepsilon}\,.$$

For ε small enough,

$$C\varepsilon e^{\zeta/\varepsilon} \ge 3n_D\psi$$

holds for $\psi \in [1,1+\beta]$. It follows that

$$G_\varepsilon^+(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi \ge n_D\psi \quad \text{ for } \psi \in [1+\zeta, 1+\beta]\,.$$

Therefore (5.8) still holds on $[1,1+\beta]$, and the result is proved.

For the estimate (5.7), we have, according to (4.10)

$$G_\varepsilon^-(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi \sim C\varepsilon e^{\psi/\varepsilon^2}\,,$$

and, therefore, for $\varepsilon \to 0$ and $0 < \psi < -\beta$,

$$Y(\varepsilon,\psi) = (G_\varepsilon^-(-\beta+\gamma\varepsilon^2,\psi) - 2n_D\psi)^{-1/2} \to 0\,.$$

Then $Y(\varepsilon,\psi)$ can be bounded by an integrable function of ψ independent of ε by the same methods as above. ■

Proposition 5.2 *Let g_0, g_1 be nonincreasing functions such that (2.22) holds. In addition let n_D be such that*

$$n_D < \pi^2/2 \tag{5.9}$$

holds. Then, for ε small enough, there exists a unique solution φ_c^ε of equation (3.23). Moreover,

$$\varphi_c^\varepsilon = -\beta + \gamma\varepsilon^2 + o(\varepsilon^2) \quad \textit{as } \varepsilon \to 0 \tag{5.10}$$

holds, where γ is such that

$$\int_0^1 \frac{d\psi}{\sqrt{2A_0 e^\gamma \sqrt{\psi} - 2n_D\psi}} = 1\,. \tag{5.11}$$

Corollary 5.3 *: For ε small enough, there exists a solution $(f^\varepsilon, \varphi^\varepsilon, n^\varepsilon)$ of problem (2.17) – (2.21), with $n_D < \pi^2/2$.*

Proof of proposition 5.2: The equation (5.11) implicitely determines γ as a function of n_D. We let

$$J(\gamma, n_D) = \int_0^1 \frac{d\psi}{\sqrt{2A_0 e^\gamma \sqrt{\psi} - 2n_D \psi}} .$$

J is defined for $\gamma \geq \gamma_0 = \log \frac{n_D}{A_0}$, is decreasing with respect to γ, and tends to 0 as $\gamma \to \infty$. Straightforward computations show

$$J(\gamma_0, n_D) = \frac{1}{\sqrt{2n_D}} \int_0^1 \frac{d\psi}{\sqrt{\sqrt{\psi} - \psi}} = \frac{\pi}{\sqrt{2n_D}} .$$

Thus, (5.11) has a solution $\gamma > \log \frac{n_D}{A_0}$ if and only if $\pi/\sqrt{2n_D} > 1$ or, equivalently, if (5.9) holds. The end of the proof then follows exactly that of proposition 4.2. ■

Proposition 5.4 *We have*

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(x) = \varphi^0(x) = -\beta + \chi^0(x) , \quad x \in (0,1) , \tag{5.12}$$

where $\chi^0(x)$ is implicitely defined by

$$\int_0^{\chi^0(x)} \frac{d\psi}{\sqrt{2A_0 e^\gamma \sqrt{\psi} - 2n_D \psi}} = x . \tag{5.13}$$

Proposition 5.5 *For any $x \in (0,1)$ we have in $\mathcal{M}_b(I\!\!R)$*

$$\lim_{\varepsilon \to 0} f^\varepsilon(x,v) = \frac{j^0}{\sqrt{\varphi^0(x) + \beta}} \delta(v - \sqrt{\varphi^0(x) + \beta}) , \tag{5.14}$$

where j^0 is determined from

$$\int_0^1 \frac{d\psi}{\sqrt{4j^0 \sqrt{\psi} - 2n_D \psi}} = 1 . \tag{5.15}$$

The proofs of propositions 5.4 and 5.5 are identical to those of propositions 4.4 and 4.5. Relation (5.15) simply follows from the fact that $j^0 = A_0 e^\gamma/2$ and from (5.11). Again, the same remark applies to the present case. Let $n^0(x)$ be the density

$$n^0(x) = \frac{j^0}{\sqrt{\varphi^0(x) + \beta}} . \tag{5.16}$$

From $\varphi^0(x) + \beta = \chi^0(x)$ and (5.13), we have

$$\frac{(\varphi^0)'}{\sqrt{4j^0\sqrt{\varphi^0+\beta}-2n_D(\varphi^0+\beta)}}=1 \tag{5.17}$$

or, by taking the square of (5.17) and differentiating again:

$$(\varphi^0)''=\frac{j^0}{\sqrt{\varphi^0+\beta}}-n_D=n^0-n_D\,, \tag{5.18}$$

meaning that φ^0 satisfies the Poisson equation with boundary data $\varphi^0(0)=-\beta, \varphi^0(1)=1-\beta$. Also, it can be proved that j^0 given by (5.15) can be characterized as the unique value of the current for which the solutions of (5.18) satisfy the additional constraint $(\varphi^0)'(0)=0$.

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