Macroscopic Models for Ionization in the Presence of Strong Electric Fields *

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Abstract. Starting from a kinetic model for electrons in a weakly ionized plasma, a spherical harmonic expansion (SHE-) model is derived by a formal asymptotic analysis for strongly disparate masses of electrons and ions. The macroscopic limit of the SHE-model for high fields gives a high field extension of standard macroscopic models for ionization.

1 Introduction

We consider the following scaled kinetic equation for electrons in a medium of ions and neutral particles, subject to ionization and recombination reactions:

$$\varepsilon^2 \frac{\partial f}{\partial t} + \varepsilon v \cdot \nabla_x f - \varepsilon E \cdot \nabla_v f \quad = \quad \varepsilon^2 \tilde{Q}_{ion}(f) + \tilde{Q}_{coll}(f) \,, \tag{1.1}$$

where $f(x,v,t)$, $x, v \in \mathbb{R}^3$, $t \in \mathbb{R}$, is the electron distribution function, $E(x,t)$ is the (given) electric field, and the dimensionless parameter $\varepsilon^2 \ll 1$ is the ratio of the electron mass to the mass of a neutral particle. The length scale has been chosen such that ε can also be interpreted as the Knudsen

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number, i.e., the ratio of the mean free path (of an electron between (non-ionizing) binary collisions with neutral particles) to the length scale. For the time scale a diffusion scaling has been chosen, which the following analysis will show to be appropriate (instead of a hydrodynamic scaling where ε instead of ε^2 appears in front of the time derivative).

The terms $\tilde{Q}_{ion}(f)$ and $\tilde{Q}_{coll}(f)$ describe the effects of ionization reactions and of the above mentioned binary collisions, respectively. A basic scaling assumption is that the mass ratio ε^2 is of the same order of magnitude as the ratio of typical time scales for collisions and ionization reactions. For the derivation of (1.1) and for further details of the scaling see [9].

Neutral particles and ions are assumed to be in local equilibrium, with the distribution functions given by the Maxwellians $f^n = M_{\rho_n,u,T}$ and $f^i = M_{\rho_i,u,T}$, respectively, with

$$M_{\rho,u,T}(v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-u|^2}{2T}\right) ,$$

and given space and time dependent $\rho_n, \rho_i, T \in \mathbb{R}$ and $u \in \mathbb{R}^3$. This can be derived [9] by assuming that the typical densities of electrons and ions are much smaller than those of neutral particles. Then, the dynamics of ions and neutral particles are dominated by binary collisions between neutral particles with each other as well as between neutral particles and ions. Therefore, both species are in local equilibrium with the same mean velocity and temperature.

We consider an impact ionization process where an highly energetic electron hits a neutral particle knocking out an additional electron and leaving back an ion. The reverse process is also considered. The ionization operator is then given by

$$\begin{aligned} \tilde{Q}_{ion}(f)(v) &= \int_{\mathbb{R}^{12}} (f_1^i f_2 f_3 - f_5^n f) d\mu_{1235} \\ &\quad +2 \int_{\mathbb{R}^{12}} (f_5^n f_4 - f_1^i f_3 f) d\mu_{1345} \, , \end{aligned} \tag{1.2}$$

with the abbreviation

$$\begin{aligned} d\mu_{1235} &= \sigma_{ion}(v_1, v_2, v_3 \to v, v_5) \delta((1-\varepsilon^2)v_1 + \varepsilon(v_2 + v_3 - v) - v_5) \\ &\quad \times \delta((1-\varepsilon^2)|v_1|^2 + |v_2|^2 + |v_3|^2 - |v|^2 - |v_5|^2 + 2\Delta) dv_1 dv_2 dv_3 dv_5 \end{aligned}$$

and a similar definition of $d\mu_{1345}$, with v_2 and v replaced by v and v_4, respectively, and dv_2 replaced by dv_4. The support of the first Delta-distribution represents momentum conservation, and the second describes the loss of energy (Δ) in the ionization process. In (1.2), the subscript j on a distribution function means evaluation at v_j. The ionization rate is assumed to be translation invariant:

$$\begin{aligned}\sigma_{ion}(v_1+\overline{v}, v_2+\varepsilon\overline{v}, v_3+\varepsilon\overline{v} \to v_4+\varepsilon\overline{v}, v_5+\overline{v}) \\ = \sigma_{ion}(v_1, v_2, v_3 \to v_4, v_5)\,,\end{aligned}$$

for every $\overline{v} \in \mathbb{R}^3$. The appearance of the parameter ε in the above equation is due to the fact that the electron velocities (v_2, v_3, v_4) have been scaled differently from the ion and neutral particle velocities (v_1, v_5).

Models of the form (1.2) for ionization and recombination phenomena are studied in [8], where an existence theorem is proven.

The binary collisions between electrons and neutral particles are modelled by

$$\tilde{Q}_{coll}(f)(v) = \sigma_{coll}\int_{\mathbb{R}^3}\int_{(v-\varepsilon v_*)\cdot\Omega>0} |v-\varepsilon v_*|(f_*^{n\prime}f' - f_*^n f)d\Omega dv_*\,, \qquad (1.3)$$

with $0 < \sigma_{coll} \in \mathbb{R}$, $|\Omega| = 1$, and

$$\begin{aligned} v' &= v - \frac{2}{1+\varepsilon^2}((v-\varepsilon v_*)\cdot\Omega)\Omega\,, \\ v_*' &= v_* + \frac{2\varepsilon}{1+\varepsilon^2}((v-\varepsilon v_*)\cdot\Omega)\Omega\,. \end{aligned} \qquad (1.4)$$

The subsequent computations are greatly simplified by the coordinate transformation $v - \varepsilon u \to v$. The transformed version of equation (1.1) is given by

$$\begin{aligned}\varepsilon^2\frac{\partial f}{\partial t} - \varepsilon^3\frac{\partial u}{\partial t}\cdot\nabla_v f + \varepsilon(v+\varepsilon u)\cdot(\nabla_x f - \varepsilon\nabla_x u\nabla_v f) - \varepsilon E\cdot\nabla_v f \\ = \varepsilon^2 Q_{ion}(f) + Q_{coll}(f)\,,\end{aligned} \qquad (1.5)$$

where $Q_{ion}(f)$ is defined by (1.2) with f^n and f^i replaced by $M_{\rho_n,0,T}$ and $M_{\rho_i,0,T}$, respectively. Similarly, $Q_{coll}(f)$ is defined by (1.3) with f^n replaced by $M_{\rho_n,0,T}$. Thus, in (1.5) the mean velocity of the background medium appears in the transport operator instead of the ionization and collision operators.

In the following section some results on the asymptotics of the collision operator Q_{coll} as $\varepsilon \to 0$ are presented. This is an extension of some of the results of [3]. Section 3 contains the formal asymptotics as $\varepsilon \to 0$ for solutions of (1.5). In the limit, the electron distribution function is approximated by an isotropic distribution satisfying a diffusion equation with both spatial diffusion as well as diffusion in the energy direction. Related results can be found in [6]. The limiting equation is an example of a SHE model, typical for situations where the dominating collision mechanism is (almost) elastic. This situation occurs frequently in semiconductor physics [1], [5], [11]. For hydrodynamic limits of disparate mass binary gases under different scaling assumptions see [4].

Finally, in section 4, a macroscopic limit of the SHE model is computed under the scaling assumption that the influence of the electric field balances that of the inelastic contributions to binary collisions. The equilibrium distribution is field dependent and interpolates between the Maxwellian distribution for low fields and the Druyvenstein distribution for low temperatures. The resulting equation for the macroscopic density contains field dependent transport parameters and ionization rates. A related treatment for the transport in semiconductors can be found in [2].

2 Properties of the collision operator

Before carrying out the limit $\varepsilon \to 0$ in (1.5) we collect some properties of the collision operator

$$Q_{coll}(f)(v) = \sigma_{coll} \int_{\mathbb{R}^3} \int_{(v-\varepsilon v_*)\cdot\Omega>0} |v-\varepsilon v_*|(f' M'_* - f M_*) d\Omega dv_* \,,$$

where here and in the following $M = M_{\rho_n,0,T}$.

Assuming a formal expansion of a solution of (1.5) in the form $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3)$, and letting $\varepsilon \to 0$ in (1.5) we obtain

$$Q_0(f_0) := \sigma_{coll}\rho_n |v| \int_{v\cdot\Omega>0} (\hat{f}_0 - f_0) d\Omega = 0 \,, \tag{2.1}$$

where $\hat{}$ denotes evaluation at $\hat{v} = v - 2(v\cdot\Omega)\Omega$, the ($\varepsilon = 0$)-version of the postcollisional velocity v'. It has been shown in [3] that the kernel of the elastic collision operator Q_0 consists of the functions depending on v only through the energy $w = |v|^2/2$. Thus, $f_0 = f_0(w)$ holds. Note that in this section we shall suppress the dependence of the distribution function on x

and t. The velocity will frequently be written in terms of energy and angular coordinates: $v = \omega\sqrt{2w}$, $\omega \in S^2$.

Another important property of Q_0 is its symmetry with respect to the $L^2(\mathbb{R}^3)$ inner product:

$$\int_{\mathbb{R}^3} Q_0(f) g \, dv = -\frac{\sigma_{coll}\rho_n}{2} \int_{\mathbb{R}^3} |v| \int_{v\cdot\Omega>0} (\hat{f} - f)(\hat{g} - g) d\Omega dv \,. \tag{2.2}$$

The collision invariants of Q_0 are all functions of the energy w, implying

$$\int_{S^2} Q_0(f) d\omega = 0 \,.$$

For an ε-independent argument f, the collision operator can be expanded in the form $Q_{coll}(f) = Q_0(f) + \varepsilon Q_1(f) + \varepsilon^2 Q_2(f) + O(\varepsilon^3)$. Then, for a solution of (1.5) we have

$$\begin{aligned} Q_{coll}(f) &= \varepsilon(Q_0(f_1) + Q_1(f_0)) \\ &\quad + \varepsilon^2(Q_0(f_2) + Q_1(f_1) + Q_2(f_0)) + O(\varepsilon^3) \,. \end{aligned}$$

It is the main objective of this section to compute information on the terms in this expansion.

First we collect some basic results, the proofs of which follow from straightforward computations:

Lemma 2.1 *i) For any $\Omega \in S^2$,*

$$\int_{\mathbb{R}^3} M_*(v_* \cdot \Omega)^2 dv_* = \rho_n T \,.$$

ii) For any $v \in \mathbb{R}^3$,

$$\int_{v\cdot\Omega>0} (v \cdot \Omega)^2 d\Omega = \frac{2\pi}{3} |v|^2 \,.$$

Lemma 2.2 *Formal computations give $Q_1(f_0) = 0$ and*

$$\begin{aligned} &\sqrt{2w} \int_{S^2} [Q_1(f_1) + Q_2(f_0)] d\omega = \sqrt{2w} \int_{S^2} Q_2(f_0) d\omega \\ &= \frac{64\pi^2}{3} \sigma_{coll} \rho_n T \frac{\partial}{\partial w} \left[w^2 \left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w} \right) \right] \,. \end{aligned}$$

Proof. We start with a weak formulation of the collision term. With a test function of the form $\varphi = \varphi_0(w) + \varepsilon\varphi_1(v)$, we have

$$\int_{\mathbf{R}^3} Q_{coll}(f)\varphi dv = \varepsilon \int_{\mathbf{R}^3} Q_1(f_0)\varphi_0 dv + \varepsilon^2 \int_{\mathbf{R}^3} [Q_0(f_1) + Q_1(f_0)]\varphi_1 dv + \varepsilon^2 \int_{\mathbf{R}^3} [Q_1(f_1) + Q_2(f_0)]\varphi_0 dv + O(\varepsilon^3)\,. \tag{2.3}$$

On the other hand, we use the symmetry property

$$\int_{\mathbf{R}^3} Q_{coll}(f)\frac{g}{M} dv = -\frac{\sigma_{coll}}{2}\int_{\mathbf{R}^6}\int_{(v-\varepsilon v_*)\cdot\Omega>0} |v-\varepsilon v_*| M M_* \times \left(\frac{f'}{M'} - \frac{f}{M}\right)\left(\frac{g'}{M'} - \frac{g}{M}\right) d\Omega dv_* dv\,,$$

with $g = g_0(w) + \varepsilon g_1(v)$. Using (1.4) we expand

$$\frac{f'}{M'} - \frac{f}{M} = \frac{\varepsilon}{M}(F_0 + F_1) + O(\varepsilon^2)\,,$$

$$F_0 = -2(v_*\cdot\Omega)(v\cdot\Omega)\left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right)\,, \qquad F_1 = \hat{f}_1 - f_1\,,$$

implying

$$\int_{\mathbf{R}^3} Q_{coll}(f)\frac{g}{M} dv = -\varepsilon^2\frac{\sigma_{coll}}{2}\int_{\mathbf{R}^6}\int_{v\cdot\Omega>0} |v|\frac{M_*}{M}(F_0 + F_1)(G_0 + G_1) d\Omega dv_* dv + O(\varepsilon^3)\,.$$

With the notation

$$A_{ij} = -\frac{\sigma_{coll}}{2}\int_{\mathbf{R}^6}\int_{v\cdot\Omega>0} |v|\frac{M_*}{M} F_i G_j d\Omega dv_* dv\,,$$

(2.2) implies

$$A_{11} = \int_{\mathbf{R}^3} Q_0(f_1)\frac{g_1}{M} dv\,.$$

Also, $A_{10} = A_{01} = 0$ holds since the Maxwellian has zero mean velocity and therefore the integral with respect to v_* vanishes. Finally, in

$$A_{00} = -2\sigma_{coll}\int_{\mathbf{R}^6} |v|\frac{M_*}{M}\left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right)\left(\frac{g_0}{T} + \frac{\partial g_0}{\partial w}\right) \int_{v\cdot\Omega>0} (v_*\cdot\Omega)^2 (v\cdot\Omega)^2 d\Omega dv_* dv\,,$$

we use Lemma 2.1:

$$A_{00} = -\frac{4\pi}{3}\sigma_{coll}\rho_n T \int_{\mathbf{R}^3} |v|^3 \left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right) \frac{\partial}{\partial w}\left(\frac{g_0}{M}\right) dv \,.$$

After the introduction of energy and angular variables and an integration by parts this becomes

$$A_{00} = \frac{64\pi^2}{3}\sigma_{coll}\rho_n T \int_0^\infty \frac{\partial}{\partial w}\left[w^2\left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right)\right] \frac{g_0}{M} dw \,.$$

Comparison of

$$\int_{\mathbf{R}^3} Q_{coll}(f) \frac{g}{M} dv = \varepsilon^2 (A_{00} + A_{11}) + O(\varepsilon^3)$$

for $g = \varphi M$ with (2.3) completes the proof. ∎

3 The diffusion approximation

In this section we carry out the formal limit $\varepsilon \to 0$ in the transport equation (1.5). Substituting the ansatz $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3)$ in (1.5) and comparing coefficients of powers of ε gives

$$\begin{aligned}
&0 = Q_0(f_0)\,, \\
&v \cdot \nabla_x f_0 - E \cdot \nabla_v f_0 = Q_0(f_1) + Q_1(f_0)\,, \\
&\frac{\partial f_0}{\partial t} + u \cdot \nabla_x f_0 - v^{tr} \nabla_x u \nabla_v f_0 + v \cdot \nabla_x f_1 - E \cdot \nabla_v f_1 \\
&\qquad = Q^0_{ion}(f_0) + Q_0(f_2) + Q_1(f_1) + Q_2(f_0)\,.
\end{aligned} \tag{3.1}$$

The ionization operator Q^0_{ion} is obtained by setting $\varepsilon = 0$ in Q_{ion}:

$$\begin{aligned}
&Q^0_{ion}(f) \\
&= \int_{\mathbf{R}^6} \sigma^0_{ion}(v_2, v_3 \to v)\delta(w_2 + w_3 + \Delta - w)(\rho_i f_2 f_3 - \rho_n f) dv_2 dv_3 \\
&+2\int_{\mathbf{R}^6} \sigma^0_{ion}(v, v_3 \to v_4)\delta(w + w_3 + \Delta - w_4)(\rho_n f_4 - \rho_i f f_3) dv_3 dv_4\,,
\end{aligned}$$

with $w = |v|^2/2$, $w_j = |v_j|^2/2$, $j = 2, 3, 4$, and

$$\sigma^0_{ion}(v_2, v_3 \to v_4) = \int_{\mathbf{R}^3} \sigma_{ion}(v_1, v_2, v_3 \to v_4, v_1) M_{1,0,T}(v_1) dv_1 \,.$$

The results of the previous section and the first equation imply $f_0 = f_0(x, w, t)$. Also, by Lemma 2.2, the second equation can be written as

$$v \cdot \left(\nabla_x f_0 - E \frac{\partial f_0}{\partial w} \right) = Q_0(f_1) \, .$$

A straightforward computation using (2.1) shows that the general solution is given by

$$f_1 = -\frac{3}{4\pi \sigma_{coll} \rho_n |v|} v \cdot \left(\nabla_x f_0 - E \frac{\partial f_0}{\partial w} \right) + \overline{f}_1 \, ,$$

with $\overline{f}_1 = \overline{f}_1(x, w, t)$. Finally, an equation for the determination of f_0 is obtained by integrating (3.1) over balls. As a preliminary computation, it is easily shown that

$$\begin{aligned} & \sqrt{2w} \int_{S^2} (u \cdot \nabla_x f_0 - v^{tr} \nabla_x u \nabla_v f_0) d\omega \\ & = \nabla_x \cdot (4\pi \sqrt{2w} \, u f_0) - \frac{\partial}{\partial w} \left(\frac{4\pi}{3} (2w)^{3/2} (\nabla_x \cdot u) f_0 \right) \end{aligned}$$

holds. Then, the above mentioned integration of (3.1) with lemma 2.2 gives

$$\begin{aligned} & \sqrt{2w} \frac{\partial f_0}{\partial t} + \nabla_x \cdot (\sqrt{2w} \, u f_0) - \frac{\partial}{\partial w} \left(\frac{1}{3} (2w)^{3/2} (\nabla_x \cdot u) f_0 \right) + \nabla_x \cdot j \\ & - \frac{\partial}{\partial w} (E \cdot j) = \frac{\partial}{\partial w} \left[\frac{16\pi \sigma_{coll} \rho_n T}{3} w^2 \left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w} \right) \right] + \overline{Q}_{ion}(f_0) \end{aligned} \qquad (3.2)$$

with

$$4\pi j := \sqrt{2w} \int_{S^2} v f_1 d\omega = -\frac{2w}{\sigma_{coll} \rho_n} \left(\nabla_x f_0 - E \frac{\partial f_0}{\partial w} \right)$$

and

$$\begin{aligned} \overline{Q}_{ion}(f) &= \frac{\sqrt{2w}}{4\pi} \int_{S^2} Q^0_{ion}(f) d\omega \\ &= \int_0^\infty \int_0^\infty \sqrt{w_2 w_3 w} \, \overline{\sigma}_{ion}(w_2, w_3 \to w) \\ & \qquad \delta(w_2 + w_3 + \Delta - w)(\rho_i f_2 f_3 - \rho_n f) dw_2 dw_3 \qquad (3.3) \\ & + 2 \int_0^\infty \int_0^\infty \sqrt{w w_3 w_4} \, \overline{\sigma}_{ion}(w, w_3 \to w_4) \\ & \qquad \delta(w + w_3 + \Delta - w_4)(\rho_n f_4 - \rho_i f f_3) dw_3 dw_4 \, , \end{aligned}$$

where

$$
\begin{aligned}
&\overline{\sigma}_{ion}(w_2, w_3 \to w_4) \\
&= \frac{1}{\pi\sqrt{2}} \int_{(S^2)^3} \sigma^0_{ion}(\omega_2\sqrt{2w_2}\,, \omega_3\sqrt{2w_3} \;\to \omega_4\sqrt{2w_4}\,) d\omega_2 d\omega_3 d\omega_4 \,.
\end{aligned}
$$

The derivation of (3.2) has been the aim of our formal asymptotics. The equation can also be written as

$$
\sqrt{2w}\frac{\partial f_0}{\partial t} + \nabla_x \cdot J_x + \frac{\partial J_w}{\partial w} = \overline{Q}_{ion}(f_0)\,, \tag{3.4}
$$

with

$$
\begin{aligned}
J_x &= \sqrt{2w}\, u f_0 - \frac{w}{2\pi\sigma_{coll}\rho_n}\left(\nabla_x f_0 - E\frac{\partial f_0}{\partial w}\right), \\
J_w &= \frac{w}{2\pi\sigma_{coll}\rho_n} E\cdot\left(\nabla_x f_0 - E\frac{\partial f_0}{\partial w}\right) - \frac{(2w)^{3/2}}{3}(\nabla_x \cdot u) f_0 \\
&\quad - \frac{16\pi\sigma_{coll}\rho_n T}{3} w^2 \left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right).
\end{aligned}
$$

Proposition 3.1 *The operator on the left hand side of (3.4) is parabolic for $w > 0$, but not uniformly parabolic.*

Proof. A simple computation shows that the leading part of the second order operator in position-energy direction is represented by the quadratic form

$$
\frac{w}{2\pi\sigma_{coll}\rho_n}|X - EW|^2 + \frac{16\pi\sigma_{coll}\rho_n T}{3} w^2 W^2\,, \qquad X \in \mathbb{R}^3,\ W \in \mathbb{R}\,,
$$

which is positive definite, but not uniformly, for $w > 0$. ■

The degeneracy of the equation at the energy boundary $w = 0$ implies that the natural zero flux condition

$$
J_w = 0\,, \qquad \text{for } w = 0 \tag{3.5}
$$

is sufficient as a boundary condition. It only excludes too strong singularities of f_0 at $w = 0$.

4 A Macroscopic Limit at High Fields

In this section we assume that ionization effects are weak compared to the inelastic contribution of the collision effects. Starting with the diffusion model (3.4), we rewrite the ionization cross section $\overline{\sigma}_{ion}$ as $\gamma\overline{\sigma}_{ion}$ with a small dimensionless parameter γ. We are interested in effects occurring on length and time scales relevant for the ionization processes. Therefore, the rescaling $x \to x/\gamma$, $t \to t/\gamma$ is introduced in (3.4):

$$\gamma\sqrt{2w}\frac{\partial f_0}{\partial t} + \gamma\nabla_x \cdot J_x + \frac{\partial J_w}{\partial w} = \gamma\overline{Q}_{ion}(f_0)\,, \tag{4.1}$$

with

$$\begin{aligned}
J_x &= \sqrt{2w}\,uf_0 - \frac{w}{2\pi\sigma_{coll}\rho_n}\left(\gamma\nabla_x f_0 - E\frac{\partial f_0}{\partial w}\right)\,, \\
J_w &= \frac{w}{2\pi\sigma_{coll}\rho_n}E\cdot\left(\gamma\nabla_x f_0 - E\frac{\partial f_0}{\partial w}\right) - \gamma\frac{(2w)^{3/2}}{3}(\nabla_x \cdot u)f_0 \\
&\quad -\frac{16\pi\sigma_{coll}\rho_n T}{3}w^2\left(\frac{f_0}{T} + \frac{\partial f_0}{\partial w}\right)\,.
\end{aligned}$$

The ionization operator $\overline{Q}_{ion}$ is given by (3.3).

In the formal limit $\gamma \to 0$, (4.1) reduces to

$$\frac{\partial}{\partial w}\left((A^2 w + Tw^2)\frac{\partial f_0}{\partial w} + w^2 f_0\right) = 0\,,$$

with

$$A(x,t) = \sqrt{\frac{3}{2}}\,\frac{|E(x,t)|}{4\pi\sigma_{coll}\rho_n(x,t)}\,.$$

The general solution satisfying the zero flux condition (3.5) is given by $f_0(x,w,t) = \rho_e(x,t)\mathcal{M}_{A(x,t),T(x,t)}(w)$ with arbitrary ρ_e and

$$\mathcal{M}_{A,T}(w) = c_{A,T}\exp\left(-\int_0^w \frac{s\,ds}{sT + A^2}\right)\,,$$

where $c_{A,T}$ is chosen such that

$$\int_0^\infty \sqrt{2w}\mathcal{M}_{A,T}(w)\,dw = 1\,.$$

Limiting cases of the equilibrium distribution are the Maxwellian

$$\mathcal{M}_{0,T}(w) = \sqrt{\frac{2}{\pi T^3}}\, e^{-w/T}$$

and the Druyvenstein distribution [7]

$$\mathcal{M}_{A,0}(w) = \frac{1}{A^{3/2}2^{1/4}\Gamma(3/4)}\, e^{-w^2/(2A^2)} .$$

An equation for the macroscopic electron density ρ_e is obtained by integrating (4.1) with respect to w:

$$\frac{\partial \rho_e}{\partial t} + \nabla_x \cdot [\rho_e(u - \mu E)] = q_{ion}(\rho_e) .$$

The mobility is given by

$$\mu = \frac{1}{2\pi\sigma_{coll}\rho_n} \int_0^\infty \mathcal{M}_{A,T}(w)dw ,$$

and the macroscopic ionization rate by

$$q_{ion}(\rho_e) = \rho_e(a\rho_n - b\rho_i\rho_e) ,$$

with

$$\begin{aligned}
a(A,T,\Delta) &= \int_0^\infty \int_0^\infty \int_0^\infty \sqrt{w_2 w_3 w_4}\,\overline{\sigma}_{ion}(w_2, w_3 \to w_4) \\
&\qquad \delta(w_2 + w_3 + \Delta - w_4)\mathcal{M}_{A,T}(w_4)dw_2 dw_3 dw_4 , \\
b(A,T,\Delta) &= \int_0^\infty \int_0^\infty \int_0^\infty \sqrt{w_2 w_3 w_4}\,\overline{\sigma}_{ion}(w_2, w_3 \to w_4) \\
&\qquad \delta(w_2 + w_3 + \Delta - w_4)\mathcal{M}_{A,T}(w_2)\mathcal{M}_{A,T}(w_3)dw_2 dw_3 dw_4 .
\end{aligned}$$

The equilibrium condition $\rho_i\rho_e = \frac{a}{b}\rho_n$ is a high field generalization of the Saha law [10]

$$\rho_i\rho_e = \sqrt{\frac{\pi T^3}{2}}\, e^{-\Delta/T}\rho_n ,$$

which is recovered for $A = 0$.

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