Existence of solutions of a kinetic equation modelling cometary flows

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Abstract

A global existence theorem is presented for a kinetic problem of the form

$$\partial_t f + v \cdot \nabla_x f = Q(f)\,,$$
$$f(t=0) = f_0\,,$$

where $Q(f)$ is a simplified model wave-particle collision operator extracted from quasilinear plasma physics. Evaluation of $Q(f)$ requires the computation of the mean velocity of the distribution f. Therefore, the assumptions on the data are such that vacuum regions, where the mean velocity is not well defined, are excluded. Also the initial data are assumed to have bounded total energy. As additional results conservation laws for mass, momentum, and energy are derived, as well as an entropy dissipation law and the propagation of higher order moments.

Key words: kinetic equation, wave-particle collision operator, cometary flows, cosmic rays, quasilinear plasma theory, approximate solution, dispersive lemma.

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1 Introduction and main results

In this work we consider a kinetic initial value problem of the form

$$\partial_t f + v \cdot \nabla_x f = Q(f)\,, \qquad (1)$$

$$f(0,x,v) = f_0(x,v)\,, \qquad (2)$$

where $f(t,x,v)$ is the particle distribution function in the position-velocity phase space $\mathbb{R}^{2d} \ni (x,v)$ at the time $t > 0$, $d \geq 1$ the dimension. The collision operator is a simplified model from quasilinear plasma theory describing wave-particle interaction in cometary flows:

$$Q(f) = P_{u_f}(f) - f\,, \qquad (3)$$

where P_u is a projection on the set of distribution functions isotropic around the velocity $u \in \mathbb{R}^d$:

$$P_u(f)(v) = \frac{1}{|\mathcal{S}^{d-1}|} \int_{\mathcal{S}^{d-1}} f(u + |v-u|\omega)\, d\omega\,, \qquad (4)$$

with $|\mathcal{S}^{d-1}|$ denoting the Lebesgue measure of the unit sphere $\mathcal{S}^{d-1}$ in $\mathbb{R}^d$. The mass, momentum, and energy densities associated with f are given by

$$\rho_f = \int_{\mathbb{R}^d} f\, dv\,, \qquad m_f = \int_{\mathbb{R}^d} f v\, dv\,, \qquad E_f = \int_{\mathbb{R}^d} f \frac{|v|^2}{2}\, dv\,. \qquad (5)$$

Finally, the mean velocity and the specific internal energy are

$$u_f = \frac{m_f}{\rho_f}\,, \qquad e_f = \frac{E_f}{\rho_f} - \frac{|u_f|^2}{2} = \frac{1}{\rho_f}\int_{\mathbb{R}^d} f \frac{|v-u_f|^2}{2}\, dv\,. \qquad (6)$$

Note the nonlinearity of Q induced by the appearance of the mean velocity u_f in the projection. The equations (1)–(6) are in dimensionless form. In particular, a relaxation time appearing in the dimensional version of the collision operator has been used as reference time.

A mathematical treatment of this model has been started in [4], [5]. More specifically, [4] was devoted to the derivation of the equations governing the macroscopic regime at the level of the Hilbert expansion. On the other hand, in [5] the results of [4] were extended by carrying out the Chapman-Enskog expansion. Also, the macroscopic behaviour for small perturbations of a global equilibrium has been analyzed in the framework of a diffusive scaling of the kinetic model.

For the physical background we refer, for example, to the series of papers [8, 14, 15, 16]. Indeed, in 1988 Earl, Jokipii and Morfill presented in [8] an 'extended transport equation' for cosmic rays including new effects due to cosmic-ray viscosity and inertia, providing the description of the evolution of the (momentum-) isotropic part of the distribution of particles with a prescribed nonrelativistic velocity. This equation was improved to include the effect of an average magnetic field embedded in the fluid, as shown in [14] and [16], and coupled with the momentum conservation equation of the fluid in [15].

The collision operator Q describes the scattering of cosmic rays (energetic particles) in an astrophysical plasma, caused by random irregularities (random spectrum of waves) in the ambient magnetic field [8]. This is the reason why we refer to Q as a wave-particle collision operator. The quasilinear theory of plasmas [13] provides complex expressions for such operators. Nevertheless, following [8] and the previous works [4], [5], we shall consider the relaxation time model (3), which — in spite of its simplicity — contains most of the fundamental features of hydrodynamics. This statement is a consequence of the formal results below.

We start by collecting some formal properties of the linear collision operator Q_u, $u \in \mathbb{R}^d$, defined by $Q_u(f) = P_u(f) - f$ (see [4], [5]):

Lemma 1 *For arbitrary $u \in \mathbb{R}^d$, $f, g \in \mathcal{D}(\mathbb{R}^d)$, $\psi \in C^\infty([0,\infty))$,*
(i) $\psi(|v-u|)$ is a collision invariant of Q_u:

$$\int_{\mathbb{R}^d} Q_u(f)(v)\psi(|v-u|)dv = 0\,, \tag{7}$$

(ii) Q_u is symmetric with respect to the $L^2(\mathbb{R}^d)$-inner product:

$$\int_{\mathbb{R}^d} Q_u(f)g\,dv = -\int_{\mathbb{R}^d} Q_u(f)Q_u(g)dv\,, \tag{8}$$

(iii) P_u has the monotonicity property

$$a \le f(v) \le b \qquad \Longrightarrow \qquad a \le P_u(f)(v) \le b\,. \tag{9}$$

Most of the main properties of the nonlinear operator Q are consequences of this result [4], [5]:

Lemma 2 *For arbitrary $f \in \mathcal{D}(\mathbb{R}^d)$ with $\rho_f > 0$, $\psi \in C^\infty([0,\infty))$,*
(i) $\psi(|v-u_f|)$ and v are collision invariants of Q:

$$\int_{\mathbb{R}^d} Q(f)(v)\psi(|v-u_f|)dv = \int_{\mathbb{R}^d} Q(f)(v)v\,dv = 0\,, \tag{10}$$

(ii) the following H-theorem holds:

$$\int_{\mathbb{R}^d} Q(f)f\,dv = -\int_{\mathbb{R}^d} Q(f)^2\,dv \le 0\,, \tag{11}$$

(iii) $Q(f)=0$ *iff there exist* $u\in\mathbb{R}^d$ *and* $F\in C^\infty([0,\infty))$*, such that* $f(v)=F(|v-u|)$*.*

Remark 1 *The statements of Lemmata 1 and 2 can be extended for less regular functions by density arguments, whenever the involved integrals are well defined. This is the way those results will be used in the following.*

A distinctive feature of the cometary flow model as compared to the classical kinetic theory of gas dynamics [3] is that the set of collision invariants (as well as the set of equilibrium distributions) is infinite dimensional and depends on the distribution function through the mean velocity. Contained in this set are 1, v, and $|v|^2 = |v-u_f|^2 + 2u_f\cdot v - |u_f|^2$, implying mass, momentum, and energy conservation. These properties are used in [4], [5] for the derivation of macroscopic limits.

In the present paper, a global existence theorem for the problem (1)–(6) is proved. Also the fundamental conservation and dissipation properties are verified rigorously.

Theorem 1 (Existence for the nonlinear problem) *Let* $f_0\in L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ *be nonnegative, satisfy* $E_{f_0}\in L^1(\mathbb{R}^d)$*, and*

$$\int_{\mathbb{R}^d} f_0(x-vt,v)dv \ge \gamma_{K,T} > 0 \qquad x\in K\,,\ t\in[0,T]\,, \tag{12}$$

for every compact $K\subset\mathbb{R}^d$ *and* $T>0$*. Then, there exists a global, nonnegative weak solution* $f\in L^\infty((0,\infty);\ L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d}))$ *of the problem* (1)–(6)*. For the mass, momentum, and energy densities given by* (5)*,*

$$\rho_f\,,\ m_f\,,\ E_f\in L^\infty((0,\infty);\ L^1(\mathbb{R}^d)) \tag{13}$$

holds. The mean velocity and specific internal energy, given by (6)*, satisfy*

$$u_f\in L^\infty_{loc}([0,\infty);\ L^2_{loc}(\mathbb{R}^d))\,,\qquad e_f\in L^\infty_{loc}([0,\infty);\ L^1_{loc}(\mathbb{R}^d))\,. \tag{14}$$

Remark 2 *Note that any continuous positive* $f_0\in L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ *satisfies assumption* (12)*. An example for an admissible initial datum is the Gaussian* $f_0(x,v)=\exp(-|x|^2-|v|^2)$*.*

Theorem 2 (Propagation of moments) *Let* f_0 *satisfy the assumptions of Theorem 1 and* $(|v|^p+|x|^q)f_0\in L^1(\mathbb{R}^{2d})$ *with* $1\le q\le p$*. Then, solutions of* (1)–(6)*, as given in Theorem 1, satisfy* $(|v|^p+|x|^q)f\in L^\infty_{loc}([0,\infty);\ L^1(\mathbb{R}^{2d}))$*.*

Theorem 3 (Conservation laws) *Let the assumptions of Theorem 1 hold. Then, the following conservation laws hold for solutions f of* (1)–(6)*:*

$$\int_{\mathbb{R}^d} \begin{pmatrix} \rho_f \\ m_f \\ E_f \end{pmatrix} dx = \int_{\mathbb{R}^d} \begin{pmatrix} \rho_{f_0} \\ m_{f_0} \\ E_{f_0} \end{pmatrix} dx \,, \tag{15}$$

$$\partial_t \begin{pmatrix} \rho_f \\ m_f \\ E_f \end{pmatrix} + \nabla_x \cdot \int_{\mathbb{R}^d} f \begin{pmatrix} v \\ v \otimes v \\ v|v|^2/2 \end{pmatrix} dv = 0 \,, \tag{16}$$

where (16) *has to be understood in the sense of distributions. We also have the entropy dissipation*

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} f^2 \, dv \, dx = -2 \int_{\mathbb{R}^{2d}} Q(f)^2 \, dv \, dx \,, \tag{17}$$

and, if $|x|^2 f_0 \in L^1(\mathbb{R}^{2d})$*,*

$$\int_{\mathbb{R}^{2d}} f|x - vt|^2 \, dv \, dx = \int_{\mathbb{R}^{2d}} f_0 |x|^2 \, dv \, dx \,. \tag{18}$$

This section is concluded by an outline of the remainder of the paper: The following section contains an extension of the definition of the collision operator to nonsmooth arguments as well as some stability properties. The proof of Theorem 1 is carried out in Section 3. It relies on the construction of an approximate solution, designed such that we can pass to the limit to obtain a solution of the original problem. For this purpose, we use a compactness argument based on the velocity averaging lemmas introduced in [9] and improved in [10]. These results have already been widely exploited to deal with existence problems for nonlinear kinetic equations as, for instance, in [6] for the purpose of establishing global existence of a solution to the Boltzmann equation, [12] and [11] for the existence of solutions to the BGK equation, or [2] for the same purpose concerning the radiative transfer equations. Section 4 is devoted to the proofs of Theorems 2 and 3.

2 Properties of the collision operator

We shall need a definition of $Q_u(f)$ for nonsmooth functions $u(t,x)$ and $f(t,x,v)$. We shall only be concerned with $P_u(f)$ in this section. However, all the results trivially carry over to $Q_u(f) = P_u(f) - f$.

Let $u: (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borelian function with $|u| < \infty$ a.e. in $(0,\infty) \times \mathbb{R}^d$ and $f \in \mathcal{D}((0,\infty) \times \mathbb{R}^{2d})$. Then $P_u(f)(t,x,v)$ is defined by (4) for every (t,x) with $|u(t,x)| < \infty$ and by $P_u(f)(t,x,v) = 0$ otherwise. An immediate consequence of this definition is the following lemma:

Lemma 3 *Let $u, \tilde{u}: (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be Borelian functions with $|u|, |\tilde{u}| < \infty$ and $u = \tilde{u}$ a.e. in $(0,\infty) \times \mathbb{R}^d$. Let $f \in \mathcal{D}((0,\infty) \times \mathbb{R}^{2d})$. Then $P_u(f) = P_{\tilde{u}}(f)$ a.e. in $(0,\infty) \times \mathbb{R}^{2d}$.*

Proof: Denote by $N \subset (0,\infty) \times \mathbb{R}^d$ the set of measure zero where $u \neq \tilde{u}$. Then $P_u(f) \neq P_{\tilde{u}}(f)$ in a subset of $N \times \mathbb{R}^d$ which is a set of measure zero in $(0,\infty) \times \mathbb{R}^{2d}$. ∎

Lemma 4 *With the assumptions of the previous lemma on u and f, with $1 \le p,q \le \infty$, and with $T > 0$, we have*

$$\|P_u(f)\|_{L^q\left((0,T);\; L^p(\mathbb{R}^{2d})\right)} \le \|f\|_{L^q\left((0,T);\; L^p(\mathbb{R}^{2d})\right)} .$$

Proof: The inequality $|P_u(f)|^p \le P_u(|f|^p)$ is easily shown by an application of the Hölder inequality for $1 < p < \infty$ and obvious for $p = 1, \infty$. Integration with respect to v gives

$$\|P_u(f)(t,x,\cdot)\|_{L^p(\mathbb{R}^d)} \le \|f(t,x,\cdot)\|_{L^p(\mathbb{R}^d)} , \tag{19}$$

implying the result. ∎

As a consequence of Lemma 4, P_u can be considered as a bounded linear operator on $L^q((0,T);\; L^p(\mathbb{R}^{2d}))$ for every Borelian function u with $|u| < \infty$ a.e. in $(0,T) \times \mathbb{R}^d$. The final result of this section is concerned with the stability of $P_u(f)$ with respect to u:

Lemma 5 *Let $f \in L^q((0,T);\; L^p(\mathbb{R}^{2d}))$ with $1 \le p,q < \infty$ and let $\lim_{n\to\infty} u_n = u$ in $L^1_{loc}((0,T) \times \mathbb{R}^d)^d$. Then*

$$\lim_{n\to\infty} P_{u_n}(f) = P_u(f) \qquad \text{in } L^q((0,T);\; L^p(\mathbb{R}^{2d})) .$$

Proof: By Lemma 4 and a density argument it is sufficient to carry out the proof for test functions $f \in \mathcal{D}((0,T) \times \mathbb{R}^{2d})$. Furthermore, since for such a test function

$$\|P_{u_n}(f)\|_{L^\infty\left((0,T)\times\mathbb{R}^{2d}\right)} \le \|f\|_{L^\infty\left((0,T)\times\mathbb{R}^{2d}\right)}$$

holds, it is sufficient to prove convergence in $L^1((0,T) \times \mathbb{R}^{2d})$.

The main difficulty of the proof results from the fact that even for a test function f, $P_u(f)$ does not necessarily have compact support if u is unbounded. In the basic estimate

$$|P_{u_n}(f)(t,x,v) - P_u(f)(t,x,v)| \le c(f)|u_n(t,x) - u(t,x)| \tag{20}$$

(which is an obvious consequence of the Lipschitz continuity of f) the constant $c(f)$ could be redefined as a function of (t,x) with compact support, but it has to be chosen independently of v in general. Therefore (20) cannot be used directly to prove the lemma.

Let $K \subset (0,T) \times \mathbb{R}^d$ be a compact set in (t,x)-space such that $\mathrm{supp}(f) \subset K \times \mathbb{R}^d$. This obviously implies

$$\mathrm{supp}(P_{u_n}(f)),\ \mathrm{supp}(P_u(f)) \subset K \times \mathbb{R}^d . \tag{21}$$

Also u_n converges to u in $L^1(K)$. Therefore a subsequence (again denoted by u_n) converges to u a.e. in K. The Egoroff theorem implies for every $\varepsilon > 0$ the existence of a set $A_\varepsilon \subset K$ with $\mathrm{meas}(K \setminus A_\varepsilon) \le \varepsilon$ such that $u_n \to u$ uniformly in A_ε. We also introduce the set

$$B_M = \{(t,x) \in K : \ |u(t,x)| < M\} .$$

It is easy to see that

$$\mathrm{meas}(K \setminus B_M) \le \frac{1}{M}\|u\|_{L^1(K)} .$$

By the uniform convergence of u_n,

$$|u_n| < 2M \qquad \text{in } A_\varepsilon \cap B_M$$

holds for n large enough. As a consequence, there exists a compact set $K_v \subset \mathbb{R}^d$ with

$$\mathrm{supp}(P_{u_n}(f)(t,x,\cdot)), \ \mathrm{supp}(P_u(f)(t,x,\cdot)) \subset K_v \tag{22}$$

for $(t,x) \in A_\varepsilon \cap B_M$. By (21) we have

$$\|P_{u_n}(f) - P_u(f)\|_{L^1\left((0,T)\times\mathbb{R}^{2d}\right)} = \int_{\mathbb{R}^d}\int_K |P_{u_n}(f) - P_u(f)| dt\, dx\, dv \le \mathcal{A} + \mathcal{B} + \mathcal{C} ,$$

where the three terms on the right hand side correspond to the splitting $K = (A_\varepsilon \cap B_M) \cup (K \setminus A_\varepsilon) \cup (K \setminus B_M)$. In the estimation of $\mathcal{A}$ we use (22) and (20):

$$\mathcal{A} \le \int_{K_v}\int_{A_\varepsilon \cap B_M} c(f)|u_n - u| dt\, dx\, dv \le c_1(f)\|u_n - u\|_{L^1(K)} .$$

For estimating $\mathcal{B}$ and $\mathcal{C}$, (19) implies

$$\begin{aligned} \mathcal{B} &\le 2\int_{\mathbb{R}^d}\int_{K\setminus A_\varepsilon} |f| dt\, dx\, dv \le c_2(f)\varepsilon , \\ \mathcal{C} &\le 2\int_{\mathbb{R}^d}\int_{K\setminus B_M} |f| dt\, dx\, dv \le c_3(u,f)\frac{1}{M} . \end{aligned}$$

Going to the limit $n \to \infty$ now gives

$$\limsup_{n\to\infty} \|P_{u_n}(f) - P_u(f)\|_{L^1\left((0,T)\times\mathbb{R}^{2d}\right)} \le c_2(f)\varepsilon + c_3(u,f)\frac{1}{M} ,$$

implying the sought for convergence result by $\varepsilon \to 0$ and $M \to \infty$.

We recall that u_n is a subsequence of the original sequence. Convergence of the full sequence, however, follows from the uniqueness of the limit, which is a consequence of Lemma 3. ∎

3 The existence result

We start with the formulation of a problem formally approximating (1)–(6). For that purpose we define, for $n \in \mathbb{N}$, the velocity truncation

$$\varphi_n(u)(t,x) = \begin{cases} u(t,x), & \text{for } |u(t,x)| < n,\ |x| < n, \\ n\frac{u(t,x)}{|u(t,x)|}, & \text{for } |u(t,x)| \geq n,\ |x| < n, \\ 0 & \text{for } |x| \geq n, \end{cases} \tag{23}$$

and consider the sequence of problems

$$\partial_t f^n + v \cdot \nabla_x f^n = Q^n(f^n)\,, \tag{24}$$

$$f^n(0,x,v) = f_0(x,v)\,, \tag{25}$$

as an approximation of (1)–(6), with

$$Q^n(f) = Q_{\varphi_n(u_f)}(f)\,. \tag{26}$$

The following existence result holds:

Proposition 1 (Existence of an approximate solution) *Let f_0 satisfy the assumptions of Theorem 1. Then, for every $n \in \mathbb{N}$, there exists a nonnegative, weak solution $f^n \in L^\infty((0,n);\ L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}))$ of (23)–(26) with $E_{f^n} \in L^\infty((0,n);\ L^1(\mathbb{R}^d))$. The bounds for f^n and E_{f^n} in the respective spaces are independent of n.*

The proof relies on a fixed-point argument based on solving linearized problems. Therefore we prove as a preliminary result existence for the linear problem with given velocity in the projection operator:

Proposition 2 (Existence and uniqueness for the linear problem) *Let f_0 satisfy the assumptions of Theorem 1 and let $u \in L^\infty((0,\infty) \times \mathbb{R}^d)^d$ with $\|u\|_{L^\infty((0,\infty)\times\mathbb{R}^d)^d} = M$. Then the problem*

$$\partial_t f + v \cdot \nabla_x f = Q_u(f)\,, \tag{27}$$

$$f(0,x,v) = f_0(x,v)\,, \tag{28}$$

has a unique nonnegative solution $f \in L^\infty((0,\infty);\ L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}))$ with

$$\|f(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}\,, \qquad \|f(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^{2d})} \leq \|f_0\|_{L^\infty(\mathbb{R}^{2d})}\,. \tag{29}$$

Moreover, $E_f \in L^\infty_{loc}([0,\infty);\ L^1(\mathbb{R}^d))$, $u_f \in L^\infty_{loc}([0,\infty);\ L^2_{loc}(\mathbb{R}^d))$ holds. The bounds on E_f and u_f in the respective spaces only depend on f_0 and M.

Sketch of the proof: Existence and uniqueness can be achieved by a simple contraction-type fixed point argument (using Lemma 4) that we omit here. The inequalities

$$0 \le f(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^{2d})}$$

follow by a standard iterative argument from the nonnegativity of f_0 and from the monotonicity property (9) of P_u. The mass conservation property (29) follows from integration of (27) with respect to v, x, and t.

In order to prove the boundedness of E_f, we use the identity

$$\int_{\mathbb{R}^d} Q_u(f)|v|^2 dv = 2u \cdot (\rho_f u - m_f)\,,$$

which follows from Lemma 1. Formally multiplying (27) by $|v|^2/2$ and integrating with respect to x and v, we get

$$\frac{d}{dt}\|E_f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} u \cdot (\rho_f u - m_f)dx \le c\left(c + \sqrt{2\|E_f\|_{L^1(\mathbb{R}^d)}}\right)$$

with

$$c = M\|f_0\|^{1/2}_{L^1(\mathbb{R}^{2d})}\,,$$

where the estimate follows from the Cauchy-Schwarz inequality

$$\|m_f\|^2_{L^1(\mathbb{R}^d)} \le 2\|\rho_f\|_{L^1(\mathbb{R}^d)}\|E_f\|_{L^1(\mathbb{R}^d)}\,. \tag{30}$$

Now the assertion of Proposition 2 on E_f is a consequence of the Gronwall lemma.

By (30), to see that u_f is well defined, we just need a lower bound on the density ρ_f. In order to find it, we use the following equivalent integral representation of (27), (28), given by Duhammel's principle:

$$f(t,x,v) = e^{-t}f_0(x - vt, v) + \int_0^t e^{s-t}P_u(f)(x - v(t-s), v, s)ds\,. \tag{31}$$

Now by (31) and assumption (12) we obtain

$$\rho_f(t,x) \ge e^{-T}\gamma_{K,T} > 0\,, \qquad \text{for } x \in K\,,\ 0 \le t \le T\,, \tag{32}$$

for every compact $K \subset \mathbb{R}^d$, and for every $T > 0$.

Combining (30) and (32) gives $u_f \in L^\infty_{loc}([0,\infty);\ L^1_{loc}(\mathbb{R}^d))$.The stronger result of Proposition 2 follows from the local-in-x-version of (30),

$$\rho_f|u_f|^2 \le 2E_f\,,$$

the bound on E_f, and (32). ∎

The next step is the solution of the approximate problem (23)–(26).

Proof of Proposition 1: We are first concerned with the existence proof. For each $n \in \mathbb{N}$ fixed, we introduce the set

$$\mathcal{S}_n = \{u \in L^1((0,n) \times B_n)^d : \ |u| \le n \text{ a.e. in } (0,n) \times B_n\} ,$$

with $B_n = \{x \in \mathbb{R}^d : \ |x| < n\}$. Then, $\mathcal{S}_n$ is a closed, convex, and bounded subset of $L^1((0,n) \times B_n)^d$. Assuming extension by zero for $x \notin B_n$, every element of $\mathcal{S}_n$ can also be considered as an element of $L^1((0,n) \times \mathbb{R}^d)^d$. With this convention, the operator φ_n, defined by (23), maps arbitrary measurable velocity fields on $(0,n) \times \mathbb{R}^d$ to $\mathcal{S}_n$.

Now an operator $T_1 : \ \mathcal{S}_n \to L^1((0,n) \times B_n)^d$ is defined in the following way: For $u \in \mathcal{S}_n$ (extended to $(0,n) \times \mathbb{R}^d$), let f denote the solution of (27)–(28), and let $T_1(u)$ be the restriction of u_f to $(0,n) \times B_n$. A fixed point operator $T : \ \mathcal{S}_n \to \mathcal{S}_n$ is then defined by $T(u) = \varphi_n(T_1(u))$. Obviously, fixed points of T correspond to solutions of (23)–(26). Since $\varphi_n : \ L^1((0,n) \times B_n)^d \to \mathcal{S}_n$ is continuous, the Schauder fixed point theorem can be applied to T, if we can prove continuity and compactness of T_1.

We first prove the compactness property. For $u \in \mathcal{S}_n$, it is a consequence of the boundedness of the solution f of (27), (28) in $L^\infty((0,\infty); \ L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}))$ that f is also bounded in $L^2((0,n) \times \mathbb{R}^{2d})$. Since Q_u is a bounded operator on this space,

$$\partial_t f + v \cdot \nabla_x f \in L^2((0,n) \times \mathbb{R}^{2d})$$

follows. Therefore, a velocity averaging lemma [10] can be applied, giving

$$\int_{K_v} f |v|^r dv \in H^{1/2}((0,n) \times \mathbb{R}^d)$$

for every compact $K_v \subset \mathbb{R}^d$. The uniform boundedness of E_f in $L^1((0,n) \times B_n)$ (see Proposition 2) implies that $\int_{\mathbb{R}^d} f|v|^r dv$ is compact in $L^1((0,n) \times B_n)$ for $0 \le r < 2$.

Thus, the map from $u \in \mathcal{S}_n$ to $\rho_f, m_f \in L^1((0,n) \times B_n)$ is compact. The lower bound

$$\rho_f \ge e^{-n} \gamma_{B_n,n} > 0 \, , \qquad \text{in } (0,n) \times B_n \, ,$$

now implies compactness of T_1.

To check its continuity, we consider a sequence $u_k \in \mathcal{S}_n$ converging to $u \in \mathcal{S}_n$ as $k \to \infty$. We denote by f_k and f the unique solutions of the linear problem (27), (28) associated with u_k and u, respectively. The uniform boundedness result from Proposition 2 implies convergence of a subsequence of f_k to $\tilde{f}$ in $L^\infty((0,\infty) \times \mathbb{R}^{2d})$ weak*. The continuity result from Lemma 5 now implies that we can go to the limit in (27), (28) in the sense of distributions. Now the uniqueness of the solution of the linear problem gives $\tilde{f} = f$ and convergence of the whole sequence f_k. Using the averaging lemma and the boundedness of ρ_{f_k} from below as above, we can go to the limit $k \to \infty$ in $u_{f_k} = m_{f_k}/\rho_{f_k}$, completing the proof of continuity of T_1. Now an application of Schauder's fixed point theorem settles the existence result.

It remains to prove the boundedness of E_{f^n}. We proceed as in the proof of Proposition 2 and integrate the product of (24) and $|v|^2$ with respect to v and x:

$$\frac{d}{dt} \|E_{f^n}\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \varphi_n(u_{f^n}) \cdot (\rho_{f^n} \varphi_n(u_{f^n}) - m_{f^n}) dx \, .$$

The observation that $\varphi_n(u)(t,x) = \theta(t,x)u(t,x)$ with $0 \le \theta \le 1$ shows that the right hand side is nonpositive, completing the proof. $\blacksquare$

Now we are ready to prove the existence theorem.

Proof of Theorem 1: We carry out the limit $n \to \infty$ in (24), (25). Extending f^n by 0 for $t \in [n,\infty)$, Proposition 1 implies that a subsequence of f^n converges to a limit f in $L^q_{loc}((0,\infty);\ L^p(\mathbb{R}^{2d});\ \text{weak})$ for every $\infty > p,q > 1$. As in the preceding proof, a velocity averaging lemma can be applied to prove the convergence (up to a subsequence) of $u_n := u_{f^n}$ to u_f in $L^1_{loc}((0,T)\times\mathbb{R}^d)$.

For a compact set $K \subset (0,T)\times\mathbb{R}^d$ we use the Egoroff theorem as in the proof of lemma 5 to deduce that (up to a subsequence) u_n converges uniformly to u_f in $A_\varepsilon \subset K$ with $\text{meas}(K \setminus A_\varepsilon) \le \varepsilon$. We also use the set

$$B_M = \{(t,x) \in K :\ |u_f(t,x)| < M\}\,,$$

as in the proof of Lemma 5. Then

$$\lim_{n\to\infty} \int_{A_\varepsilon \cap B_M} |\varphi_n(u_n) - u_f| dt\, dx = 0\,,$$

since $\varphi_n(u_n) = u_n$ on $A_\varepsilon \cap B_M$ for n large enough. On the other hand,

$$\begin{aligned}
&\int_{K\setminus(A_\varepsilon\cap B_M)} |\varphi_n(u_n) - u_f| dt\, dx \le \int_{K\setminus(A_\varepsilon\cap B_M)} (|u_n| + |u_f|) dt\, dx \\
&\overset{n\to\infty}{\longrightarrow}\quad 2\int_{K\setminus(A_\varepsilon\cap B_M)} |u_f| dt\, dx \quad \overset{M\to\infty,\ \varepsilon\to 0}{\longrightarrow} \quad 0\,,
\end{aligned}$$

implying convergence of $\varphi_n(u_n)$ to u_f in $L^1_{loc}((0,T)\times\mathbb{R}^d)$.

Thus, by the continuity results in Lemmas 4, 5, we can go to the limit in (24), (25) in the distributional sense. The bounds for the moments and for the mean velocity are obtained by going to the limit in the corresponding inequalities for the approximating problem. $\blacksquare$

4 Propagation of moments and conservation laws

For the proofs of Theorems 2 and 3 we shall need the following technical result:

Lemma 6 *For any $p \ge 1$ and for any nonnegative function f with $(1+|v|^p)f \in L^1(\mathbb{R}^d)$, $\rho_f > 0$, we have*

$$\begin{aligned}
&\textit{(i)} \qquad \rho_f |u_f|^p \le \int_{\mathbb{R}^d} |v|^p f\, dv\,, \\
&\textit{(ii)} \qquad \int_{\mathbb{R}^d} |v|^p P_{u_f}(f) dv \le C_p \int_{\mathbb{R}^d} |v|^p f\, dv\,,
\end{aligned}$$

for some positive constant C_p depending only on p.

Proof: (i) The Hölder inequality for the measure $f\,dv$ gives

$$\rho_f|u_f| \le \int_{\mathbb{R}^d} |v| f\,dv \le \rho_f^{1/p'} \left(\int_{\mathbb{R}^d} |v|^p f\,dv\right)^{1/p} ,$$

which is equivalent to the result (p' is the conjugate exponent of p).
(ii) In the following, C denotes various constants depending only on p. For $f \in \mathcal{D}(\mathbb{R}^d)$ with $\rho_f > 0$, we have

$$|v|^p \le C\left(|v-u_f|^p + |u_f|^p\right) ,$$

implying

$$\begin{aligned} \int_{\mathbb{R}^d} |v|^p P_{u_f}(f) dv &\le C\left(\int_{\mathbb{R}^d} |v-u_f|^p f\,dv + \rho_f|u_f|^p\right) \\ &\le C\left(\int_{\mathbb{R}^d} |v|^p f\,dv + \rho_f|u_f|^p\right) . \end{aligned}$$

Here, the conservation property (7) has been applied, being justified since f is smooth and has compact support. The proof is completed by an application of (i). ∎

Now the statement that $1, v, |v|^2$ are collision invariants of Q can be made rigorous:

Corollary 1 *For a solution f of* (1)–(6) *as given in Theorem 1, we have*

$$\int_{\mathbb{R}^d} Q(f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0 .$$

With the help of Lemma 6, the boundedness of higher order moments can be proved:

Proof of Theorem 2: We present a formal proof which can be completed by an appropriate smoothing. Setting $g = (1 + |x|^q + |v|^p)f$, we have

$$\partial_t g + v\cdot\nabla_x g + g = S = q(v\cdot x)|x|^{q-2} f + (1 + |x|^q + |v|^p) P_{u_f}(f) . \tag{33}$$

With

$$|v\cdot x|\,|x|^{q-2} \le |v|\,|x|^{q-1} \le \frac{|v|^q}{q} + \frac{|x|^q}{q'} ,$$

(where q' is the conjugate exponent of q) and with Lemma 6 we obtain

$$\int_{\mathbb{R}^d} |S| dv \le C \int_{\mathbb{R}^d} g\,dv .$$

Integration of (33) with respect to v and x now gives

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} g\,dv\,dx \le C\int_{\mathbb{R}^{2d}} g\,dv\,dx ,$$

and an application of the Gronwall lemma completes the proof. ∎

In the proof of Theorem 3 a classical dispersive lemma due to Perthame [12] is used, which we recall for the sake of completeness:

Lemma 7 *Assume $g_0 \in L^1(\mathbb{R}^{2d})$ and $h \in L^\infty((0,T);\ L^1(\mathbb{R}^{2d}))$. Then, the solution g of*

$$\partial_t g + v \cdot \nabla_x g = h\,, \qquad g(t=0) = g_0$$

satisfies $(1+|v|)g \in L^1((0,T) \times K_x \times \mathbb{R}^d)$ for every compact set K_x of $\mathbb{R}^d$.

Proof of Theorem 3: Let us focus on the conservation of energy. The other conservation laws are obtained similarly. In the weak formulation of (1), (2),

$$\int_0^\infty \int_{\mathbb{R}^{2d}} f\left(\partial_t \phi + v \cdot \nabla_x \phi + Q_{u_f}(\phi)\right) dv\, dx\, dt = \int_{\mathbb{R}^{2d}} f_0 \phi(t=0) dv\, dx\,,$$

we set $\phi(t,x,v) = |v|^2 \varphi(x/R)\varphi(v/V)\theta(t)$ with $\theta \in C_0^\infty([0,\infty))$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\varphi(y) = 1$ for $|y| \le 1$. The bounds of Theorem 1 justify letting $R \to \infty$:

$$\int_0^\infty \int_{\mathbb{R}^{2d}} \left(f|v|^2\varphi(v/V)\theta'(t) + Q(f)|v|^2\varphi(v/V)\theta(t)\right) dv\, dx\, dt = \theta(0)\int_{\mathbb{R}^{2d}} f_0|v|^2\varphi(v/V) dv\, dx\,.$$

In the further limit $V \to \infty$ the second term on the left hand side vanishes by Corollary 1, and total energy conservation (15) follows.

Now we remark that

$$\int_{\mathbb{R}^d} v|v|^2 f\, dv \in L^1_{loc}([0,\infty) \times \mathbb{R}^d)\,.$$

This is a consequence of Lemma 7. Choosing now a test function of the form $\phi(t,x,v) = |v|^2\varphi(v/V)\eta(t,x)$ with $\eta \in C_0^\infty([0,\infty)\times\mathbb{R}^d)$, and letting $V \to \infty$, we obtain the local version of conservation of energy.

For proving the entropy dissipation result (17), we note that $f \in L^\infty((0,\infty);\ L^2(\mathbb{R}^{2d}))$ holds. With the help of Lemma 4, this is sufficient for proving that the H-theorem (11) holds. Let us now multiply the transport equation (1) by $2f$. Since f and $\partial_t f + v \cdot \nabla_x f$ belong to $L^\infty((0,\infty);\ L^2(\mathbb{R}^{2d}))$, we have

$$(\partial_t f + v \cdot \nabla_x f)2f = \partial_t f^2 + v \cdot \nabla_x f^2\,.$$

This can be easily justified by using, for instance, a convolution by an approximation of unity and Friedrichs Lemma (see [1], also [7] and the notion of renormalized solution). Now integration with respect to v and x gives (17).

Finally, (18) is a consequence of the identity

$$|x-vt|^2(\partial_t f + v \cdot \nabla_x f) = \partial_t(|x-vt|^2 f) + v \cdot \nabla_x(|x-vt|^2 f)$$

and of the fact that $|x-vt|^2 = |x|^2 - 2t(x\cdot v) + t^2|v|^2$ is a collision invariant of Q. ∎

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