THE TRANSIENT BEHAVIOUR OF MULTI-DIMENSIONAL PN-DIODES IN LOW INJECTION

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Abstract. An initial-boundary value problem modeling the transient behaviour of a *pn*-junction semiconductor diode is simplified by formal asymptotics, where the scaled minimal Debye length and the scaled intrinsic number are considered as small parameters. The implicit assumption on the biasing situation is that of low injection. The simplified model is shown to be equivalent to an integral relation between the evolution of the current and of the contact voltage.

A simple switching application is governed by a nonlinear Volterra integral equation for the current. This equation is shown to possess a unique solution globally in time converging to the unique steady state.

1. Introduction. We consider a k-dimensional ($k = 1, 2$ or 3) model for the time dependent behaviour of a semiconductor *pn*-diode. A two-dimensional cross section of such a device is depicted in Figure 1. Our model is an initial-boundary value problem for the classical drift-diffusion equations for the flow of negatively charged electrons (density $n(\mathbf{x}, t)$) and positively charged holes (density $p(\mathbf{x}, t)$) in a semiconductor. In scaled form it consists of the continuity equations

$$\nabla \cdot \mathbf{J}_n - \frac{1}{\delta^4}\frac{\partial n}{\partial t} = R\,, \qquad -\nabla \cdot \mathbf{J}_p - \frac{1}{\delta^4}\frac{\partial p}{\partial t} = R\,, \tag{1.1a}$$

the current relations

$$\delta^4 \mathbf{J}_n = \mu_n(\nabla n - n\nabla V)\,, \qquad \delta^4 \mathbf{J}_p = -\mu_p(\nabla p + p\nabla V)\,, \tag{1.1b}$$

and the Poisson equation

$$\lambda^2 \Delta V = n - p - C \tag{1.1c}$$

for the electrostatic potential $V(\mathbf{x}, t)$. The mobilities $\mu_n, \mu_p > 0$ and the doping profile C are assumed to be given functions of position $\mathbf{x} \in \Omega$, where the bounded domain $\Omega \subset \mathbb{R}^k$ represents the semiconductor part of the device.

Figure 1 should be placed here.

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We assume the Shockley-Read-Hall term

$$R = \frac{np/\delta^4 - 1}{\tau_p(n+\delta^2) + \tau_n(p+\delta^2)}\,,$$

for recombination-generation via a trap in the middle of the forbidden band. Since this work is restricted to low injection situations, our model certainly describes the relevant physical phenomena [9].

The equations (1.1) are in dimensionless form. The reference quantity for the particle densities n, p, C is the maximal doping concentration C_{max}, i.e. $\max_\Omega |C| = 1$ holds. The potential has been scaled by the thermal voltage $U_T = kT/q$ where k, T and q denote the Boltzmann constant, the lattice temperature and the elementary charge, respectively. The reference length L is the diameter of the device and, thus, $\mathrm{diam}(\Omega) = 1$. The mobilities are scaled by a characteristic value $\widetilde{\mu}$ and the reference time is given by the diffusion time $L^2/(\widetilde{\mu} U_T)$. Finally, the reference value

$$\frac{q\widetilde{\mu} U_T}{L} \frac{n_i^2}{C_{max}} \tag{1.2}$$

for the electron and hole current densities $\mathbf{J}_n$ and $\mathbf{J}_p$, respectively, contains the intrinsic density n_i. In low injection situations the factor n_i^2/C_{max} is a typical minority carrier density. Also it is well known that in low injection the currents are governed by the flow of minority carriers. Thus, (1.2) is a characteristic value for current densities driven by a field of the strength U_T/L. For further details on the scaling of the drift-diffusion equations we refer the reader to [8] and [9].

The equations (1.1) contain the dimensionless parameters

$$\lambda = \frac{1}{L}\sqrt{\frac{\varepsilon U_T}{q C_{max}}}\,, \qquad \delta^2 = \frac{n_i}{C_{max}}\,,$$

which can be interpreted as scaled versions of the minimal Debye length and of the intrinsic number, respectively.

The semiconductor domain Ω can be split into the n-region Ω_n and the p-region Ω_p separated by the pn-junction Γ, a smooth $(k-1)$-dimensional manifold (see Figure 1). The (simply connected) n- (p-)region is characterized by the fact that the doping profile is positive (negative) there. We assume an abrupt junction, i.e. the doping profile has a jump discontinuity along Γ and is bounded away from zero within the n- and p-regions:

$$C \geq \gamma > 0\,, \quad \text{in } \Omega_n\,, \qquad C \leq -\gamma < 0\,, \quad \text{in } \Omega_p$$

The boundary $\partial\Omega$ of the device is assumed to be Lipschitz. It is the disjoint union of the Ohmic contacts $\Gamma_n \subset \partial\Omega \cap \partial\Omega_n$ and $\Gamma_p \subset \partial\Omega \cap \partial\Omega_p$ and artificial or insulating boundary segments $\partial\Omega_N$. At the Ohmic contacts we assume zero space charge and thermal equilibrium:

$$n - p - C = 0\,, \quad np = \delta^4\,, \qquad \text{at } \Gamma_n \cup \Gamma_p\,,$$

which translates to Dirichlet boundary conditions for the charge carrier densities:

$$n = \frac{1}{2}\left(C + \sqrt{C^2 + 4\delta^4}\right), \quad p = \frac{1}{2}\left(-C + \sqrt{C^2 + 4\delta^4}\right), \qquad \text{at } \Gamma_n \cup \Gamma_p \tag{1.3a}$$

For the potential, we have the boundary conditions

$$V = V_{bi} - U(t), \quad \text{at } \Gamma_n, \qquad V = V_{bi}, \quad \text{at } \Gamma_p, \tag{1.3b}$$

where $U(t)$ is the external voltage between the contacts, and the built-in potential is given by

$$V_{bi} = \ln \frac{C + \sqrt{C^2 + 4\delta^4}}{2\delta^2}. \tag{1.3c}$$

Along the artificial and insulating boundary segments, we assume that the normal components of the electric field and of the electron and hole current densities vanish. This amounts to homogeneous Neumann conditions for V, n and p:

$$\frac{\partial V}{\partial \boldsymbol{\nu}} = \frac{\partial n}{\partial \boldsymbol{\nu}} = \frac{\partial p}{\partial \boldsymbol{\nu}} = 0, \qquad \text{at } \partial\Omega_N, \tag{1.3d}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal.

The formulation of an initial-boundary value problem is completed by prescribing initial conditions for the carrier densities:

$$n(\mathbf{x}, 0) = n_0(\mathbf{x}), \qquad p(\mathbf{x}, 0) = p_0(\mathbf{x}) \tag{1.4}$$

We shall assume that the initial data n_0 and p_0 correspond to a steady state solution of (1.1), (1.3) with the initial contact voltage U_0 used in (1.3b).

The current $I(t)$ through the device can be computed as an integral over one of the contacts or — alternatively — over the pn-junction:

$$I(t) = \int_\Gamma \mathbf{J}_{tot} \cdot \boldsymbol{\nu}\, ds,$$

where $\boldsymbol{\nu}$ denotes the unit normal vector along Γ pointing into Ω_n and the total current density

$$\mathbf{J}_{tot} = \mathbf{J}_n + \mathbf{J}_p - \frac{\lambda^2}{\delta^4}\nabla\frac{\partial V}{\partial t}$$

is given as the sum of the particle current densities and the density of the displacement current.

The dimensionless parameters λ and δ^2 are small compared to 1 in practical applications. Our approach is an asymptotic analysis where we let these parameters tend to zero consecutively.

The limit $\lambda \to 0$ has been carried out by Ringhofer [12], [13]. It is reviewed in section 2. Setting $\lambda = 0$ corresponds to the physical assumption of zero space charge. Essentially, it amounts to replacing the left hand side of the Poisson equation (1.1c) by zero. However, since the problem is singularly perturbed in terms of λ, layer behaviour has to be expected. In particular, the jump discontinuities of the doping profile cause the limiting potential and the carrier densities to have jumps across the pn-junction. Also, an initial jump has to be expected for the time dependent problem. The nature of these jumps can be analyzed by introducing slow variables which are continuous in the limit.

The time dependent problem becomes a parabolic-algebraic system in the limit. In the language of the theory of differential-algebraic equations [3], it is an index 2 problem [1]. This means that the initial conditions have to satisfy certain compatibility relations which is guaranteed by the assumption that the initial data originate from a steady state solution. In other words this assumption implies the absence of initial jumps [12].

The small parameter λ also appears in the expression for the total current density in front of the displacement current density. One of the simplifications introduced by our analysis is that displacement current effects are neglected.

As opposed to the stationary problem, where convergence of solutions to solutions of the zero space charge problem as $\lambda \to 0$ can be proven [5], [14], a rigorous theory justifying the limiting procedure is not available for the transient case.

A further simplification of the problem is introduced by letting δ^2 tend to zero. Keeping the applied voltages fixed as the built-in potential tends to infinity (as $\delta^2 \to 0$, see (1.3b), (1.3c)) can be interpreted as a low injection condition.

For steady state problems this limit has been formally carried out in [9] and rigorously justified in [14]. However, the zero space charge and low injection assumptions have already been used for one-dimensional model problems in the early physical literature on semiconductor devices [6], [7], [16]. In particular, the famous Shockley equation [16] for the voltage-current characteristic of a pn-diode is derived in this way. The present work can be seen as an extension of the results by Kingston [6] and Lax and Neustadter [7] on the switching behaviour.

For one-dimensional pn-diodes with infinitely long n- and p-regions the limit $\delta^2 \to 0$ in the zero space charge problem has been carried out in [15]. Here we extend these results to multi-dimensional devices in section 2. The limiting problem consists of linear parabolic equations in the n- and p-regions coupled by interface conditions at the pn-junction. We show in section 3 that this problem is equivalent to an integral equation of convolution type describing the evolution of the current in terms of the evolution of the contact voltage. The kernel of the integral equation is given in the form of an infinite series with terms determined by the solution of an eigenvalue problem.

In section 4, we consider the simple switching circuit depicted in Figure 2. At time $t = 0$ the switch S is thrown to the right and we are interested in the decay to the new steady state. In terms of our simplified model, the current solves a nonlinear Volterra integral equation of the second kind. We prove existence and uniqueness of a solution for all times, as well as convergence to the steady state determined from the Shockley equation for the voltage current characteristic.

Figure 2 should be placed here.

A short discussion of the classical model problem of Kingston [6], Lax and Neustadter [7] is contained in section 5. Here we consider a one-dimensional, strongly asymmetrically doped, infinitely long diode. A detailed analysis of this special case can be found in [15].

2. Asymptotics. The limit of the equations (1.1) as $\lambda \to 0$ constitutes a differential-algebraic system in time. If the algebraic solution component V (no time derivatives of V appear in the equations) could be determined from the algebraic equation

$$0 = n - p - C\,, \tag{2.1}$$

the system would be of index 1 [3]. Since this is not the case we are confronted with a system of higher index. As usual in the analysis of differential algebraic equations, we differentiate (2.1) with respect to time and obtain from (1.1a)

$$\nabla \cdot (\mathbf{J}_n + \mathbf{J}_p) = 0\,. \tag{2.2}$$

Since, after substitution of (1.1b), this can be interpreted as an elliptic equation for V, we obtain an index 1 system replacing (2.1) by (2.2). Note that (2.1) remains satisfied in this case as long as it is satisfied initially. The original system is of index 2 in the language of the theory of differential algebraic equations, as has been pointed out by Ascher [1]. Initial data for the problem have to be compatible with both (2.1) and (2.2). It has already been observed [12], [17] that initial layers of rapid variation occur if the initial data do not satisfy (2.1), (2.2). For the steady state solutions used as initial conditions in this work, these conditions are certainly true and, thus, no initial layers are present.

For the further analysis it is convenient to introduce the new variables w and $\widetilde{V}$ by

$$np = \delta^4 w\,, \qquad \widetilde{V} = V - \widetilde{V}_{bi}\,,$$

where

$$\widetilde{V}_{bi} = \begin{cases} \ln(C/\delta^2)\,, & \text{in } \Omega_n, \\ -\ln(|C|/\delta^2)\,, & \text{in } \Omega_p, \end{cases}$$

is an approximation for the built-in potential for small δ^2: $V_{bi} = \widetilde{V}_{bi} + O(\delta^4)$

The limiting equations for $\lambda \to 0$ in terms of w and $\widetilde{V}$ can be written as

$$\begin{aligned} &\nabla \cdot \mathbf{J}_n = -\nabla \cdot \mathbf{J}_p = \frac{1}{n+p}\frac{\partial w}{\partial t} + \frac{w-1}{\tau_p(n+\delta^2) + \tau_n(p+\delta^2)}\,, \\ &\nabla w = p\mathbf{J}_n/\mu_n - n\mathbf{J}_p/\mu_p\,, \\ &(n+p)\nabla\widetilde{V} = \nabla C\left(1 - \frac{n+p}{|C|}\right) - \delta^4(\mathbf{J}_n/\mu_n + \mathbf{J}_p/\mu_p) \end{aligned} \tag{2.3}$$

with the carrier densities given by

$$n = \frac{1}{2}\left(C + \sqrt{C^2 + 4\delta^4 w}\right)\,, \qquad p = \frac{1}{2}\left(-C + \sqrt{C^2 + 4\delta^4 w}\right)\,. \tag{2.4}$$

The boundary conditions at the Ohmic contacts for the new variables read

$$\begin{aligned} &w = 1\,, \quad \widetilde{V} = -U(t) + O(\delta^4)\,, \qquad \text{at } \Gamma_n\,, \\ &w = 1\,, \quad \widetilde{V} = O(\delta^4)\,, \qquad \text{at } \Gamma_p\,. \end{aligned}$$

Jump conditions at the pn-junction have to be considered. In [12] it has been shown that the quasi Fermi levels φ_n and φ_p, related to the densities and the potential by

$$n = \delta^2 e^{V-\varphi_n}\,, \qquad p = \delta^2 e^{\varphi_p - V}\,,$$

as well as the normal components of the current densities remain continuous across the junction in the limit. Therefore also $w = \exp(\varphi_p - \varphi_n)$ is continuous.

Continuity of the electron quasi Fermi level at Γ implies the equation

$$n_- e^{-V_-} = n_+ e^{-V_+}\,, \qquad \text{on } \Gamma,$$

where the subscripts "+" and "−" refer to one-sided limits from the Ω_n- and the Ω_p-side, respectively. In terms of the new variables w and $\widetilde{V}$ this can be written as

$$w = \frac{1}{4} e^{\widetilde{V}_- - \widetilde{V}_+} \left(1 + \sqrt{1 + 4\delta^4 w / C_+^2}\right)\left(1 + \sqrt{1 + 4\delta^4 w / C_-^2}\right), \qquad \text{on } \Gamma\,.$$

Now we perform the limit $\delta^2 \to 0$. From (2.4) we obtain

$$n = \max(0, C)\,, \qquad p = \max(0, -C)$$

and, in particular, $n + p = |C|$ in the limit. Therefore the last equation in (2.3) implies that $\widetilde{V}$ takes values independent of position in the n- and p-regions, i.e.

$$\widetilde{V} = -U(t)\,, \quad \text{in } \Omega_n, \qquad \widetilde{V} = 0\,, \quad \text{in } \Omega_p.$$

The limiting w satisfies

$$\frac{\partial w}{\partial t} = \begin{cases} C\nabla \cdot \left(\frac{\mu_p}{C}\nabla w\right) - (w-1)/\tau_p, & \text{in } \Omega_n, \\ |C|\nabla \cdot \left(\frac{\mu_n}{|C|}\nabla w\right) - (w-1)/\tau_n, & \text{in } \Omega_p. \end{cases} \tag{2.5}$$

The equation at the pn-junction reduces to

$$w = e^{U(t)}\,, \qquad \text{on } \Gamma\,. \tag{2.6}$$

For the current densities we have

$$\mathbf{J}_p = -\frac{\mu_p}{C}\nabla w\,, \quad \text{in } \Omega_n, \qquad \mathbf{J}_n = \frac{\mu_n}{|C|}\nabla w\,, \quad \text{in } \Omega_p\,.$$

Now it becomes apparent why we chose to compute the current through the device as an integral along the pn-junction. Our asymptotics only provides expressions for the minority carrier current densities so far. The pn-junction is the only location where the total current density is known. For the computation of the flow of the majority carriers the asymptotics would have to be carried to higher order terms. The current $I(t) = I[w](t)$ is given in terms of the functional $I[\,\cdot\,]$ defined by

$$I[w] = \int_\Gamma \left[\left(\frac{\mu_n}{|C|} \nabla w \right)_- - \left(\frac{\mu_p}{C} \nabla w \right)_+ \right] \cdot \boldsymbol{\nu} \, ds \,. \tag{2.7}$$

If the limits $\lambda \to 0$ and $\delta^2 \to 0$ are also performed in the steady state problem defining the initial conditions, then the initial datum $w_0(\mathbf{x})$ for w is the solution of a stationary version of (2.5), (2.6) with $U(t)$ replaced by the initial voltage U_0 in (2.6). The new variable

$$z = w - w_0$$

solves the equations

$$\frac{\partial z}{\partial t} = \begin{cases} C \nabla \cdot \left(\frac{\mu_p}{C} \nabla z \right) - z/\tau_p, & \text{in } \Omega_n, \\ |C| \nabla \cdot \left(\frac{\mu_n}{|C|} \nabla z \right) - z/\tau_n, & \text{in } \Omega_p, \end{cases} \tag{2.8}$$

subject to the auxiliary conditions

$$\begin{aligned} &z = 0 \,, \quad \text{on } \Gamma_n \cup \Gamma_p \,, \qquad z(\mathbf{x}, 0) = 0 \,, \\ &z = e^{U(t)} - e^{U_0} \,, \quad \text{on } \Gamma \,, \end{aligned} \tag{2.9}$$

and homogeneous Neumann conditions along $\partial \Omega_N$. Note that the values of z on Γ are the only inhomogeneities in (2.8), (2.9).

In terms of the variable z the current is given by

$$I(t) = I[z](t) + I[w_0] \,. \tag{2.10}$$

3. An integral equation for the current vs. the voltage. In this section we show that (2.8), (2.9), (2.10) is equivalent to an integral equation relating the evolution of the contact voltage $U(t)$ and of the current $I(t)$ through a pn-diode.

We consider the linear operator $\widetilde{\mathcal{A}}: H^2_\partial \to L^2(\Omega_0)$, $\Omega_0 = \Omega_n \cup \Omega_p = \Omega \setminus \Gamma$, with domain

$$H^2_\partial = \Big\{ z \in H^2(\Omega_0) \;|\; z = 0 \text{ on } \Gamma_n \cup \Gamma_p, \ \frac{\partial z}{\partial \boldsymbol{\nu}} = 0 \text{ on } \partial \Omega_N, \\ z = \text{const on } \Gamma, \ I[z] = 0 \Big\} ,$$

whose action on a function $z \in H^2_\partial$ is defined by the right hand side of (2.8). It is easy to see that $\widetilde{\mathcal{A}}$ is symmetric with respect to the $L^2(\Omega_0)$-inner product

$$\langle z_1, z_2 \rangle = \int_{\Omega_n} \frac{z_1 z_2}{C} d\mathbf{x} + \int_{\Omega_p} \frac{z_1 z_2}{|C|} d\mathbf{x} \,. \tag{3.1}$$

Theorem 3.1. *a) The operator $\widetilde{\mathcal{A}}$ has a self-adjoint extension $\mathcal{A}\colon D(\mathcal{A}) \to L^2(\Omega_0)$ with*

$$D(\mathcal{A}) \subset H^1_\partial = \{z \in H^1(\Omega_0) \mid \ z = 0 \text{ on } \Gamma_n \cup \Gamma_p, \ z = \text{const on } \Gamma\}.$$

b) There exists an orthonormal basis of $L^2(\Omega_0)$ consisting of eigenfunctions φ_k, $k = 1, 2, \cdots$, of $\mathcal{A}$. The corresponding eigenvalues λ_k, $k = 1, 2, \cdots$, are negative and have $-\infty$ as their only accumulation point.

Proof. The weak formulation of the problem of finding $z \in H^2_\partial$ with $\widetilde{\mathcal{A}} z = f$ for $f \in L^2(\Omega_0)$ reads

$$a(z, \varphi) = -\langle f, \varphi \rangle \tag{3.2}$$

for all test functions φ, where

$$\begin{aligned} a(z, \varphi) = &\int_{\Omega_n} \left(\frac{\mu_p}{C} \nabla z \cdot \nabla \varphi + \frac{z\varphi}{\tau_p C} \right) d\mathbf{x} \\ &+ \int_{\Omega_p} \left(\frac{\mu_n}{|C|} \nabla z \cdot \nabla \varphi + \frac{z\varphi}{\tau_n |C|} \right) d\mathbf{x} \end{aligned}$$

holds. We consider $a(\,\cdot\,,\,\cdot\,)$ as a bilinear form on the closure H^1_∂ of H^2_∂ in $H^1(\Omega_0)$. Obviously, H^1_∂ equipped with the $H^1(\Omega_0)$-norm is a Hilbert space. It is easy to see that a is symmetric, continuous and coercive. Thus, for any $f \in L^2(\Omega_0)$ by the Lax-Milgram lemma there exists a unique $z \in H^1_\partial$ such that (3.2) holds for all $\varphi \in H^1_\partial$. The set of all such z is denoted by $D(\mathcal{A})$ and the operator $\mathcal{A}$ is defined by

$$\langle \mathcal{A} z, \varphi \rangle = -a(z, \varphi), \qquad \text{for all } \varphi \in H^1_\partial .$$

Obviously, $\mathcal{A}$ is an extension of $\widetilde{\mathcal{A}}$. By the symmetry of a it is a symmetric operator on the Hilbert space $L^2(\Omega_0)$ equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and its range is $L^2(\Omega_0)$. Therefore [11, theorem VIII.3] $\mathcal{A}$ is self-adjoint, which proves part a) of the theorem.

Part b) is proved in the usual way by applying standard results [11, theorems VI.15, VI.16] for compact, self-adjoint operators to the inverse of $\mathcal{A}$. ∎

The next result shows that the steady state voltage-current characteristic is given by the Shockley equation [16].

Theorem 3.2. *For the initial current $I[w_0] = I_s(e^{U_0} - 1)$ holds, with*

$$\frac{1}{I_s} = \sum_{k=1}^{\infty} \frac{\Phi_k^2}{-\lambda_k},$$

where Φ_k is the constant value of φ_k at Γ, $k = 1, 2, \cdots$.

Remark. This result has already been derived in [14]. However, there the saturation current I_s has been computed in a different way.

Proof. The function $v = w_0 - 1$ vanishes at the contacts, takes the value $e^{U_0} - 1$ at the pn-junction and satisfies homogeneous Neumann conditions along $\partial\Omega_N$ and the differential equations

$$\nabla \cdot \left(\frac{\mu_p}{C}\nabla v\right) = \frac{v}{\tau_p C}\,, \qquad \text{in } \Omega_n,$$

$$\nabla \cdot \left(\frac{\mu_n}{|C|}\nabla v\right) = \frac{v}{\tau_n |C|}\,, \qquad \text{in } \Omega_p.$$

It is easy to see that there exists a unique $v \in H^1(\Omega_0)$ satisfying the above conditions. Obviously, $I[w_0] = I[v]$ holds. A straightforward computation shows that the Fourier series of v in terms of the eigenfunctions φ_k is given by

$$v = I[v] \sum_{k=1}^{\infty} \frac{\Phi_k}{-\lambda_k} \varphi_k\,. \tag{3.3}$$

A norm, equivalent to the $H^1(\Omega_0)$-norm, is given by

$$\|\varphi\|_{\mathcal{A}}^2 = a(\varphi,\varphi) = \sum_{k=1}^{\infty} (-\lambda_k) \langle \varphi, \varphi_k\rangle^2\,.$$

Thus, we have

$$\|v\|_{\mathcal{A}}^2 = \sum_{k=1}^{\infty} \frac{\Phi_k^2}{-\lambda_k}$$

proving the convergence of the series on the right hand side also appearing in the definition of the saturation current. Finally, the Shockley equation for the voltage-current characteristic is obtained by evaluation of (3.3) at Γ. ∎

The Fourier coefficients in the representation

$$z(\mathbf{x},t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k(\mathbf{x}) \tag{3.4}$$

of the solution of (2.8), (2.9) satisfy the initial value problems

$$\dot z_k = \lambda_k z_k + \Phi_k I[z](t)\,, \quad z_k(0) = 0\,, \qquad k = 1, 2, \cdots\,.$$

With the solutions of these problems and with (2.9), evaluation of (3.4) on Γ gives

$$e^{U(t)} - e^{U_0} = \int_0^t \widetilde K(t-s) I[z](s)\,ds\,, \qquad \widetilde K(t) = \sum_{k=1}^{\infty} \Phi_k^2 e^{\lambda_k t}\,. \tag{3.5}$$

By the estimate

$$\int_0^t \widetilde K(s)ds = \sum_{k=1}^{\infty} \frac{\Phi_k^2}{-\lambda_k}\left(1 - e^{\lambda_k t}\right) < \frac{1}{I_s} = \int_0^\infty \widetilde K(t)dt$$

the kernel $\widetilde{K}$ is integrable and we introduce the normalized version

$$K(t) = I_s \widetilde{K}(t) .$$

Using (2.10) and the Shockley equation in (3.5) gives

$$\int_0^t K(t-s) I(s)\, ds = I_s(e^{U(t)} - 1) - I_s(e^{U_0} - 1) \int_t^\infty K(s)\, ds . \tag{3.6}$$

Equation (3.6) is the main result of this section. It provides the desired relation between the evolution of the current $I(t)$ and the contact voltage $U(t)$. The integral term on the left hand side accounts for the influence of the history of the evolution, i.e. in particular for charge-storage effects. The exponentially decaying kernel $K(t)$ has an integrable singularity at $t = 0$ ($K(t) = O(t^{-1/2})$ for the model problem of section 5).

Instead of solving the eigenvalue problem for the operator $\mathcal{A}$, the kernel can also be determined from a Green's function[1] Z solving a modified version of problem (2.8), (2.9) where the last condition in (2.9) is replaced by

a) Z is independent from the position along Γ and

b) $I[Z](t) = \delta(t)$.

The kernel is then given in terms of the values of Z on Γ:

$$K(t) = I_s Z|_\Gamma$$

Note the important fact that the voltage-current relation can be determined completely in terms of the solution of a device dependent reference problem.

As expected, (3.6) has the property that $U(t)$ converges as $t \to \infty$ if $I(t)$ also converges. For the statement of the corresponding result we introduce the convolution

$$(\varphi * \psi)(t) := \int_0^t \varphi(t-s)\psi(s) ds .$$

For further reference we note that the convolution is a commutative as well as associative operation.

Lemma 3.3. *Let* $\sup_{t\in[0,\infty)} |\varphi(t)| \le M < \infty$ *and* $\lim_{t\to\infty} \varphi(t) = \varphi_\infty$ *be satisfied. Then*

$$\lim_{t\to\infty} (K * \varphi)(t) = \varphi_\infty$$

holds.

Proof. In the equation

$$K * \varphi - \varphi_\infty = K * (\varphi - \varphi_\infty) + \varphi_\infty (K * 1 - 1) ,$$

[1] We are indebted to Pierre Degond for pointing out this fact.

the second term on the right hand side converges to zero since, by definition, $\lim_{t\to\infty}(K * 1)(t) = 1$. In the first term we split the integral into two parts and obtain

$$|K * (\varphi - \varphi_\infty)(t)| \le 2M \int_0^\tau K(t-s)ds + \int_\tau^t K(t-s)|\varphi(s) - \varphi_\infty| ds\,.$$

It is easy to see that the second term can be made arbitrarily small by choosing τ large enough. Since the first term tends to zero for $t \to \infty$ and fixed τ the proof is completed. ■

For given contact voltage, (3.6) is a Volterra integral equation of the first kind for $I(t)$. It is well known that this problem is mathematically ill posed. Jumps in the voltage, for example, correspond to singularities in the current. Also the purely voltage controlled problem is not very sensible from a physical point of view. The effect of a serial resistance always has to be taken into account. These observations motivate the considerations of the following section.

4. A switching application. The simple switching circuit depicted in Figure 2 containing the serial resistance R has already been considered in [15]. For $t < 0$ we assume a steady state with the contact voltage $U = U_0$ and the corresponding current $I_0 = I_s(e^{U_0} - 1)$. At time $t = 0$ the switch S is suddenly thrown to the right. For positive t the relation

$$V = U(t) + R\,I(t)$$

holds. If this is substituted in (3.6), we obtain a nonlinear Volterra integral equation of the second kind for $I(t)$:

$$K * I = I_s(e^{V-RI} - 1) - I_0(1 - K * 1)\,. \tag{4.1}$$

For this equation the following result can be proven:

Theorem 4.1. *Equation (4.1) has a unique solution $I \in C[0,\infty) \cap C^\infty(0,\infty)$ converging as $t \to \infty$ to the unique solution I_∞ of*

$$I_\infty = I_s(e^{V-RI_\infty} - 1)\,.$$

For the proof of theorem 4.1 we need two auxiliary results. First we collect properties of the kernel in the integral equation.

Lemma 4.2. *The kernel K satisfies*
a) $K \in L^1(0,\infty)$ and $\lim_{t\to\infty}(K * 1)(t) = 1$.
b) K is completely monotone:

$$K \in C^\infty(0,\infty) \qquad \text{and} \qquad (-1)^j \frac{d^j K}{dt^j} \ge 0 \quad \text{for } j = 0, 1, \cdots$$

Proof. Part a) has already been proven in the preceding section. For proving part b) we define the partial sums

$$K_n(t) = I_s \sum_{k=1}^n \Phi_k^2 e^{\lambda_k t}$$

of the series defining K. Since $K_n(t)$ decays to zero as $t \to \infty$, the following holds: For every positive M and every $n \in \mathbb{N}$ there exists a nonnegative $t_0 = t_0(M, n)$ such that

$$K_n(t) \leq M\,, \quad \text{for } t \geq t_0\,, \qquad K_n(t) > M\,, \quad \text{for } 0 < t < t_0\,, \tag{4.2}$$

holds. The estimate

$$1 > (K_n * 1)(t_0) > Mt_0$$

immediately implies $t_0 < 1/M$ and, thus, considering (4.2),

$$K_n(t) < \frac{1}{t}\,, \qquad \text{for } t > 0\,.$$

From this inequality we conclude that the series defining $K(t)$ converges uniformly in compact subsets of the complex half-plane $\{t \mid \Re(t) > 0\}$. This and the analyticity of the K_n show that $K(t)$ is analytic for $t > 0$. Finally, the complete monotonicity of $K(t)$ follows because the terms in the series are completely monotone. ∎

The following lemma contains general results on Volterra integral equations. Part a) is a classical result [18]. A proof of part b) has been given by Friedman [2] (see also [4]).

Lemma 4.3. *a) The solution x of the equation $x + k * x = f$ is given by $x = f - r * f$, where the resolvent kernel r is the solution of $r + k * r = k$.*
b) If the kernel k is completely monotone, then the resolvent kernel r defined above has the same property.
c) If, additionally,

$$k \in L^1(0, \infty) \qquad \text{with } \lim_{t\to\infty} (k * 1)(t) = k_\infty\,,$$

then

$$r \in L^1(0, \infty) \qquad \text{with } \lim_{t\to\infty} (r * 1)(t) = \frac{k_\infty}{1 + k_\infty}\,.$$

Proof of c). The convolution of the equation for r with 1 gives

$$0 < r * 1 = k * 1 - k * r * 1 < k * 1$$

because of the positivity of r and k. This implies the integrability of r. The second statement is obtained by letting $t \to \infty$ in the above equation. ∎

Proof of theorem 4.1. In terms of the new variable

$$x(t) = \exp\big[R\big(I_\infty - I(t)\big)\big] - 1$$

equation (4.1) can be written as

$$x = -AK * \ln(x + 1) + B(1 - K * 1)\,, \tag{4.3}$$

with the constants A and B given by

$$A = \frac{1}{R(I_\infty + I_s)} > 0\,, \qquad B = \frac{I_0 - I_\infty}{I_s + I_\infty} > -1\,.$$

Note that $I = I_\infty$ corresponds to $x = 0$. So one of the things we intend to prove is that $x(t)$ tends to zero as $t \to \infty$.

A standard contraction mapping argument proves existence and uniqueness of a solution $x \in C[0,T]$ of (4.3) with T small enough. The regularity results of Miller and Feldstein [10] imply that the solution is infinitely differentiable away from $t = 0$ on its maximal interval of existence. Existence for all times will be deduced from an a priori estimate derived below.

We apply lemma 4.3 with $k = AK$ to the rewritten version

$$x + AK * x = AK * [x - \ln(x+1)] + B(1 - K * 1)$$

of (4.3), where the right hand side is considered as the inhomogeneity. The fact that $x - \ln(x+1) \geq 0$ for $x > -1$ holds, will be of importance in the following. The lemma implies that the resolvent kernel R, determined by $R + AK * R = AK$ is completely monotone and satisfies

$$R * 1 < \frac{A}{A+1} = \lim_{t\to\infty} (R * 1)(t) . \tag{4.4}$$

From part a) of the lemma we obtain

$$\begin{aligned} x &= AK * [x - \ln(x+1)] + B(1 - K * 1) \\ &\quad - R * AK * [x - \ln(x+1)] - B(R * 1 - R * K * 1) \\ &= R * [x - \ln(x+1)] + B\left(1 - \frac{A+1}{A} R * 1\right) . \end{aligned}$$

The positivity of R implies the estimate

$$x \geq B\left(1 - \frac{A+1}{A} R * 1\right) > -1 .$$

Using this in (4.3), the upper bound

$$x \leq -AK * \ln\left[B\left(1 - \frac{A+1}{A} R * 1\right) + 1\right] + B(1 - K * 1)$$

follows. With these a priori estimates at hand, a standard argument proves the existence of the solution for all $t > 0$. Since by (4.4) and by lemma 3.3 both the upper and the lower bound tend to zero for $t \to \infty$, the proof is complete. ■

5. A one-dimensional model problem. Here we restrict our attention to a classical model problem [6], [7], where the kernel $K(t)$ in (3.6) can be computed explicitely.

Starting out from (2.8), (2.9), (2.10) we first assume that the doping profile, the mobilities and the life times are constant in the n- and p-regions. Thus, we have

$$C = C_n , \quad \text{in } \Omega_n , \qquad C = -C_p , \quad \text{in } \Omega_p ,$$

and the differential equations (2.8) have constant coefficients.

Our next assumption is that of strongly asymmetric doping:

$$C_p \ll C_n \tag{5.1}$$

It is well known that in such an n^+p-diode only the contribution of the electrons to the total current is essential. This can be seen from the definition (2.7) of the current functional. By (5.1), the second term in the integral in (2.7) is negligible compared to the first.

We also consider a one-dimensional geometry where the pn-junction is located at $x = 0$, the p-region corresponds to positive values of x and the n-region to negative values of x. The current functional is then given by

$$I[w] = -\frac{\mu_n}{C_p}\frac{\partial w}{\partial x}(0+) .$$

This also shows that the voltage-current relation only depends on the subproblem of (2.8), (2.9) corresponding to the p-region. Finally we assume that the p-region is infinitely long. More precisely, the p-region is considered long compared to the diffusion length (see [15]). The initial current is then computed by solving the problem

$$0 = \mu_n \frac{\partial^2 w_0}{\partial x^2} - \frac{w_0 - 1}{\tau_n} ,$$

subject to the boundary conditions

$$w_0(0) = e^{U_0} , \qquad w_0(\infty) = 1 .$$

The current $I[w_0]$ can be computed explicitely. Comparison with the Shockley equation (theorem 3.2) gives the saturation current

$$I_s = \frac{1}{C_p}\sqrt{\frac{\mu_n}{\tau_n}} . \tag{5.2}$$

The problem (2.8), (2.9) becomes

$$\frac{\partial z}{\partial t} = \mu_n \frac{\partial^2 z}{\partial x^2} - \frac{z}{\tau_n} , \qquad \text{for } x > 0 ,$$

subject to the auxiliary conditions

$$z(0,t) = e^{U(t)} - e^{U_0} , \qquad z(\infty, t) = z(x, 0) = 0 .$$

A straightforward application of the Cosine-Fourier-transform leads to equation (3.6) with the saturation current (5.2) and the kernel function

$$K(t) = \frac{\exp(-t/\tau_n)}{\sqrt{\pi \tau_n t}} . \tag{5.3}$$

Note that in section 3 the domain Ω was assumed to be bounded. Therefore, the results of that section cannot be applied to the present case. However, it is easy to see that the kernel (5.3) has the properties stated in the lemmata 3.3 and 4.2 and that theorem 4.1 also holds when (5.3) is used in the integral equation (4.1). For additional qualitative results about this problem we refer the reader to [15].

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FIGURE CAPTIONS

Fig. 1: Two-dimensional cross section of a pn-diode.

Fig. 2: Switching circuit for a pn-diode.