3 Electrostastics

The Coulomb law

Experimental evidence shows that material can have a property called *electrical charge* existing in two versions, called *positive* and *negative*. Two positive charges repell each other, and the same is true for two negative charges, whereas two opposite charges attract each other. The strength of the attracting or repelling force is proportional to the product of the two charges and inversely proportional to the square of the distance between the charges. These observations are collected in the *Coulomb law*

$$F_{12} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{|x_1 - x_2|^2} \frac{x_1 - x_2}{|x_1 - x_2|}$$

for the force, which a charge Q_2 at position x_2 exerts on a charge Q_1 at position x_1. The notation $1/(4\pi\varepsilon_0)$ for the proportionality constant is chosen for later convenience. The information, if the charges are positive or negative is contained in the signs of Q_1 and Q_2. Note that the dimension of charge is determined by choosing the dimension of the proportionality constant ε_0, called the *permittivity*.

The force exerted by N charges $Q_1, \ldots, Q_N$ at positions $x_1, \ldots, x_N$ on a test charge q at position x is then given by

$$F(x) = \frac{q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i(x - x_i)}{|x - x_i|^3} \,.$$

The vector field $E(x) = F(x)/q$, i.e., the force per unit charge, is called the *electric field*.

Our next aim is to pass from point charges as above to a continuous charge distribution. For this purpose the space is split into small subdomains V_j, $j \in \mathbb{N}$ with volume ΔV_j, such that $\mathbb{R}^3 = \cup_{j=1}^{\infty} V_j$ and $V_j \cap V_k = \{\}$ for $j \neq k$. Then all $x_i \in V_j$ are approximated by $\xi_j \in V_j$, and the formula for the electric field becomes

$$E(x) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{\infty} \frac{x - \xi_j}{|x - \xi_j|^3} \sum_{x_i \in V_j} Q_i \,.$$

The inner sum is the total charge contained in the subdomain V_j. Representing the charge distribution by a charge density $\varrho(x)$, it can be approximated by

$$\sum_{x_i \in V_j} Q_i \approx \varrho(\xi_j)\Delta V_j \,,$$

whence the expression for the electric field becomes a Riemann sum. In the continuous limit $\Delta V_j \to 0$ we obtain

$$E(x) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \frac{\varrho(\xi)(x-\xi)}{|x-\xi|^3} d\xi \,.$$

This is the continuum version of the Coulomb law. A simple computation shows that the electric field is irrotational, i.e., it satisfies

$$\nabla \times E = 0 \,. \tag{3.1}$$

This implies that E can be written as a gradient. Actually $E = -\nabla\varphi$ holds with the *electrostatic potential*

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \frac{\varrho(\xi)}{|x-\xi|} d\xi \,.$$

It is a basic result from the theory of partial differential equations that $-1/(4\pi|x|)$ is a fundamental solution of the Laplace operator in $\mathbb{R}^3$. This implies the *Poisson equation*

$$-\varepsilon_0 \Delta\varphi = \varrho \,,$$

or, written as an equation for the electric field,

$$\varepsilon_0 \nabla \cdot E = \varrho \,. \tag{3.2}$$

Moving charges

Assume a dynamic situation with a time dependent charge density $\varrho(x,t)$. If no charges are created or lost, then the total charge in any control volume $\Omega \subset \mathbb{R}^3$ can only change by a flux through the boundary $\partial\Omega$. Writing the flux per unit area as the normal (to $\partial\Omega$) component of an *electric current density* $j(x,t)$, we arrive at the equation

$$\frac{d}{dt}\int_\Omega \varrho(x,t)dx = -\int_{\partial\Omega} j(x,t)\cdot\nu(x)d\sigma(x) \,,$$

with the outward unit normal $\nu(x)$ and the surface element $d\sigma(x)$. Using the divergence theorem, the surface integral on the right hand side can be written as a volume integral. Then the arbitrariness of Ω implies the *charge continuity equation*

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot j = 0\,. \tag{3.3}$$

Stationary magnetic fields

Experiments also show another force connected to the presence of electrical charges. It only occurs for moving charges. Consider a long straight wire (direction $n \in \mathbb{R}^3$, $|n| = 1$) carrying a constant *electric current* $I \geq 0$ (dimension: charge/time). A moving (with velocity v) point charge q at position x in the neighbourhood of the wire then experiences a force F with the following properties: The magnitude $|F|$ of the force is proportional to

- the speed $|v|$ of the point charge,
- the charge q,
- the current I, and
- the inverse $1/|x^\perp|$ of the distance of the point charge from the wire, where $x^\perp$ is the vector from the nearest point on the wire to x.

The direction of F is determined by the requirements that

- it is orthogonal to v and
- it lies in the plane spanned by n and $x^\perp$.

These requirements lead to the formula

$$F = \frac{\mu_0}{2\pi} \frac{qI}{|x^\perp|^2} v \times (n \times x^\perp)\,.$$

Finally, the experiments show that the proportionality constant (written as $\mu_0/(2\pi)$ for later convenience) is positive ($\mu_0 > 0$). The part of the information in F, only depending on the position of the point charge is the *magnetic field*

$$B = \frac{\mu_0}{2\pi} \frac{I(n \times x^\perp)}{|x^\perp|^2}\,. \tag{3.4}$$

The above formula gives the magnetic field produced by the electric current in the whole wire. In the following we shall generalize it to a situation with an arbitrary current density distribution. For an arbitrary point ξ on the wire we observe the geometric relations

$$n \times x^{\perp} = n \times (x - \xi)\,, \quad s = (x - \xi) \cdot n\,, \quad s^2 + |x^{\perp}|^2 = |x - \xi|^2\,, \qquad (3.5)$$

where s is the signed distance between the nearest point to x on the wire and ξ. The idea is to split the wire into small pieces of length Δs and to write the magnetic field as a sum of contributions originating from these pieces. For these contributions we make the ansatz

$$\Delta B = \frac{\mu_0}{2\pi} I n \times (x - \xi) f(|x - \xi|^2) \Delta s\,.$$

Then, in the limit $\Delta s \to 0$, the function f has to be chosen such that

$$B = \frac{\mu_0}{2\pi} I n \times (x - \xi) \int_{-\infty}^{\infty} f(s^2 + |x^{\perp}|^2) ds$$

gives the result (3.4). The vector product is outside the integral since it is independent of ξ by (3.5). A straightforward integration shows that the choice $f(z) = 1/(2z^{3/2})$ does the job. This way of writing the magnetic field allows for a straightforward generalization to a wire which is not straight but has the form of a general curve C:

$$B = \frac{\mu_0}{4\pi} I \int_C \frac{n(\xi) \times (x - \xi)}{|x - \xi|^3} ds(\xi)\,.$$

Now $n(\xi)$ is the normalized tangent vector at the point ξ on the curve, and $ds(\xi)$ is the length element.

Even more generally, me may think of a tube filled with wires with orthogonal cross section R such that the current is given by a surface integral of the current density $j = |j| n$:

$$I = \int_R |j| d\sigma$$

Using this in the computation of the magnetic field and combining the surface integral with the line integral to obtain a volume integral, we arrive at the *Biot-Savart law*

$$B(x) = \frac{\mu_0}{4\pi} \int_{\mathbf{R}^3} \frac{j(\xi) \times (x - \xi)}{|x - \xi|^3} d\xi\,.$$

This formula we rewrite as

$$B(x) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \nabla_x \frac{1}{|x-\xi|} \times j(\xi) d\xi = \nabla \times A\,,$$

with the *vector potential*

$$A(x) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{j(\xi)}{|x-\xi|} d\xi\,.$$

This immmediately implies

$$\nabla \cdot B = 0\,. \tag{3.6}$$

We again use the occurrence of the fundamental solution of the Laplacian to deduce

$$-\Delta A = \mu_0 j\,,$$

where the Laplacian on the left hand side is applied componentwise to the vector field A.

In all this we assume a stationary charge distribution satisfying $\frac{\partial \varrho}{\partial t} = 0$ and, thus, by the charge continuity equation (3.3), $\nabla \cdot j = 0$. Then the vector potential satisfies

$$\begin{aligned}
\nabla \cdot A(x) &= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} j(\xi) \cdot \nabla_x \frac{1}{|x-\xi|} d\xi = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} j(\xi) \cdot \nabla_\xi \frac{1}{|x-\xi|} d\xi \\
&= -\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \nabla \cdot j(\xi) \frac{1}{|x-\xi|} d\xi = 0\,.
\end{aligned}$$

In this computation the symmetry of $1/|x-\xi|$ with respect to its arguments x and ξ was used as well as an integration by parts. Now the vector identity $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A$ can be used to derive the *Ampère law*

$$\nabla \times B = \mu_0 j\,. \tag{3.7}$$

It should be noted that a vector potential A' is not determined uniquely by the equation $B = \nabla \times A'$. By $\nabla \times \nabla \chi = 0$, there is an arbitrary additive gradient field: $A' = A + \nabla \chi$. The choice of χ is called a *gauge*. The *Coulomb gauge* used here is characterized by the side condition $\nabla \cdot A = 0$.

The force $F = qv \times B(x)$ caused by the magnetic field acting on a charge q at position x and moving with velocity v, is called the *Lorentz force*.

4 Special relativity

The Galilean invariance of space-time contradicts experimental evidence that the speed of light is the same in different reference frames. The theory of relativity is a rather straightforward replacement of Galilean invariance by something else (*Lorentz invariance*) such that this evidence is respected.

As a first step it is assumed that the transformation to a new reference frame (coordinates (x', y', z', t')) moving with speed v in the x-direction compared to the old reference frame (coordinates (x, y, z, t)) is described by a linear transformation

$$x' = \alpha(x - vt)\,, \qquad y' = y\,, \qquad z' = z\,, \qquad t' = \beta x + \gamma t\,,$$

leaving the y- and z-directions uneffected. Three more requirements will determine the transformation uniquely:

- Volumes in space-time are not changed by the transformation.
- In the same way the origin $x' = 0$ of the new frame moves with speed v in the old frame (i.e., $x = vt$), the origin $x = 0$ in the old frame moves with speed $-v$ in the new frame (i.e., $x' = -vt'$).
- The speed of light c is invariant under the transformation.

The first condition leads to the requirement that the determinant of the Jacobian is equal to one: $\alpha(\gamma + \beta v) = 1$. By setting $x = 0$ and $x' + vt' = 0$ in the transformation, it is easily seen that the second condition requires $\alpha = \gamma$. Finally, $x = ct$ and $x' = ct'$ hold at the same time, iff $\alpha(c - v) - c(\beta c + \gamma) = 0$ holds. These equations uniquely determine the *Lorentz transformation*

$$x' = \gamma(x - vt)\,, \qquad t' = \gamma(t - vx/c^2)\,, \qquad \text{with } \gamma = (1 - v^2/c^2)^{-1/2}\,. \tag{4.8}$$

The existence of the *Lorentz factor* γ requires one of the basic facts of special relativity: Nothing can be faster than the speed of light: $|v| \leq c$.

It is obvious that, if the relative speed v of the reference frames is small compared to the speed of light, the Lorentz transformation can be approximated by the Galilean transformation

$$x' = x - vt\,, \qquad t' = t\,.$$

5 Electrodynamics – the Maxwell Equations

The aim of this section is the formulation of a Lorentz invariant theory for the dynamic behaviour of electric charges. We start by examining the charge continuity equation (3.3):

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot j = 0\,.$$

Considering a Lorentz transformation (4.8), the partial derivatives transform according to

$$\frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right)\,, \qquad \frac{\partial}{\partial x} = \gamma \left(\frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right)\,.$$

In terms of the new space-time variables, the continuity equation then becomes

$$\gamma \frac{\partial}{\partial t'} \left(\varrho - \frac{v}{c^2} j_1 \right) + \gamma \frac{\partial}{\partial x'} (j_1 - v\varrho) + \frac{\partial j_2}{\partial y'} + \frac{\partial j_3}{\partial z'} = 0\,.$$

With the transformation rule

$$\varrho' = \gamma \left(\varrho - \frac{v}{c^2} j_1 \right)\,, \qquad j_1' = \gamma(j_1 - v\varrho)\,, \qquad j_2' = j_2\,, \qquad j_3' = j_3\,, \tag{5.9}$$

the continuity equation is *Lorentz invariant*, i.e., invariant under Lorentz transformations:

$$\frac{\partial \varrho'}{\partial t'} + \nabla' \cdot j' = 0\,,$$

with $\nabla' = \nabla_{x'}$. Note that the 4-vector (ϱ, j) transforms in the same way as (t, x).

Now we start with the equations for the electric and magnetic fields in stationary ($\partial\varrho/\partial t = 0$) situations (3.1), (3.2), (3.6), (3.7):

$$\varepsilon_0 \nabla \cdot E = \varrho\,, \qquad \nabla \times B = \mu_0 j\,, \tag{5.10}$$

$$\nabla \times E = 0\,, \qquad \nabla \cdot B = 0\,. \tag{5.11}$$

The second of the inhomogeneous equations (5.10) cannot remain valid in an instationary situation since $\nabla \cdot j = -\partial\varrho/\partial t \neq 0$ contradicts the vector identity $\nabla \cdot (\nabla \times B) = 0$. Combining the Poisson equation and the continuity equation gives

$$\nabla \cdot \left(j + \varepsilon_0 \frac{\partial E}{\partial t} \right) = 0\,.$$

Thus, the *displacement current density* $\varepsilon_0 \partial E/\partial t$ provides a correction to obtain a divergence free quantity. The *inhomogeneous Maxwell equations*

$$\varepsilon_0 \nabla \cdot E = \varrho\,, \qquad \nabla \times B = \mu_0 j + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \tag{5.12}$$

are candidates for a nonstationary version of (5.10) compatible with the continuity equation. We carry out a Lorentz transformation in the first equation and in the first component of the second equation:

$$\begin{aligned} \gamma \frac{\partial E_1}{\partial x'} - \frac{\gamma v}{c^2} \frac{\partial E_1}{\partial t'} + \frac{\partial E_2}{\partial y'} + \frac{\partial E_3}{\partial z'} &= \frac{\varrho}{\varepsilon_0}\,, \\ \frac{\partial B_3}{\partial y'} - \frac{\partial B_2}{\partial z'} &= \mu_0 j_1 + \varepsilon_0 \mu_0 \gamma \left(\frac{\partial E_1}{\partial t'} - v \frac{\partial E_1}{\partial x'} \right)\,. \end{aligned}$$

We rewrite this system by first eliminating $\partial E_1/\partial t'$ and then $\partial E_1/\partial x'$:

$$\begin{aligned} \frac{\partial E_1}{\partial x'} + \gamma \frac{\partial}{\partial y'} \left(E_2 - \frac{v}{\varepsilon_0 \mu_0 c^2} B_3 \right) + \gamma \frac{\partial}{\partial z'} \left(E_3 + \frac{v}{\varepsilon_0 \mu_0 c^2} B_2 \right) &= \frac{\gamma}{\varepsilon_0} \left(\varrho - \frac{v}{c^2} j_1 \right)\,, \\ \gamma \frac{\partial}{\partial y'} (B_3 - \varepsilon_0 \mu_0 v E_2) - \gamma \frac{\partial}{\partial z'} (B_2 + \varepsilon_0 \mu_0 v E_3) &= \mu_0 \gamma (j_1 - v \varrho) + \varepsilon_0 \mu_0 \frac{\partial E_1}{\partial t'}\,. \end{aligned}$$

The second and third component of the second equation in (5.12) give

$$\begin{aligned} \frac{\partial B_1}{\partial z'} - \gamma \frac{\partial}{\partial x'} (B_3 - \varepsilon_0 \mu_0 v E_2) &= \mu_0 j_2 + \varepsilon_0 \mu_0 \gamma \frac{\partial}{\partial t'} \left(E_2 - \frac{v}{\varepsilon_0 \mu_0 c^2} B_3 \right)\,, \\ \gamma \frac{\partial}{\partial x'} (B_2 + \varepsilon_0 \mu_0 v E_3) - \frac{\partial B_1}{\partial y'} &= \mu_0 j_3 + \varepsilon_0 \mu_0 \gamma \frac{\partial}{\partial t'} \left(E_3 + \frac{v}{\varepsilon_0 \mu_0 c^2} B_2 \right)\,. \end{aligned}$$

With the transformation rules (5.9) and

$$E_1' = E_1\,,\ E_2' = \gamma \left(E_2 - \frac{v B_3}{\varepsilon_0 \mu_0 c^2} \right)\,,\ E_3' = \gamma \left(E_3 + \frac{v B_2}{\varepsilon_0 \mu_0 c^2} \right)\,, \tag{5.13}$$

$$B_1' = B_1\,,\ B_2' = \gamma (B_2 + \varepsilon_0 \mu_0 v E_3)\,,\ B_3' = \gamma (B_3 - \varepsilon_0 \mu_0 v E_2)\,, \tag{5.14}$$

the inhomogeneous Maxwell equations are obviously Lorentz invariant.

We still have to find instationary versions of the homogeneous equations (5.11). We start by rewriting the second component of the curl of E:

$$(\nabla \times E)_2 = \frac{\partial E_1}{\partial z'} - \gamma \frac{\partial E_3}{\partial x'} + \frac{\gamma v}{c^2} \frac{\partial E_3}{\partial t'} = (\nabla' \times E')_2 + \frac{\gamma v}{\varepsilon_0 \mu_0 c^2} \frac{\partial B_2}{\partial x'} + \frac{\gamma v}{c^2} \frac{\partial E_3}{\partial t'}$$

Now the relations between partial derivatives are used to replace the derivative with respect to x' by derivatives with respect to t and t'. This gives

$$(\nabla \times E)_2 + \frac{1}{\varepsilon_0 \mu_0 c^2} \frac{\partial B_2}{\partial t} = (\nabla' \times E')_2 + \frac{1}{\varepsilon_0 \mu_0 c^2} \frac{\partial B_2'}{\partial t'} + \frac{\gamma v}{c^2} \left(\frac{1}{\varepsilon_0 \mu_0 c^2} - 1 \right) E_3 \,.$$

The term on the left hand side is therefore Lorentz invariant iff

$$\varepsilon_0 \mu_0 c^2 = 1$$

holds, which we shall assume from now on. This suggests the *homogeneous Maxwell equations*

$$\nabla \cdot B = 0 \,, \qquad \nabla \times E = -\frac{\partial B}{\partial t} \tag{5.15}$$

as the instationary version of (5.11). Computations similar to the above show the Lorentz invariance of this system with the transformation rules

$$\begin{aligned} E_1' &= E_1 \,, \quad E_2' = \gamma(E_2 - vB_3) \,, \quad E_3' = \gamma(E_3 + vB_2) \,, \\ B_1' &= B_1 \,, \quad B_2' = \gamma \left(B_2 + \frac{v}{c^2} E_3 \right) \,, \quad B_3' = \gamma \left(B_3 - \frac{v}{c^2} E_2 \right) \,, \end{aligned}$$

for the electric and magnetic fields. As a final result we collect the *Maxwell equations* of electrodynamics (5.12), (5.15):

$$\begin{aligned} \varepsilon_0 \nabla \cdot E = \varrho \,, \qquad & \nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} \,, \\ \nabla \cdot B = 0 \,, \qquad & \nabla \times E = -\frac{\partial B}{\partial t} \,. \end{aligned}$$