Convergence to global equilibrium for a kinetic fermion model

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Abstract

We study the long-time asymptotics of a kinetic models for fermions in a box with periodic boundary conditions. An entropy dissipation approach is used to prove decay to the global equilibrium for this nonlinear equation, that lacks dissipation in the position variable. We prove convergence at an algebraic rate depending on the smoothness of the solution. The result relies on some initial bounds and a uniform boundedness assumption for spatial derivatives of the solution.

Key words: Kinetic equations, Fermions, Fermi-Dirac-Distribution, semi-conductors, H-theorem, relative entropy, entropy dissipation, long time asymptotics

AMS subject classification: 82C10, 35B40, 82D37, 82B10, 82C21, 35Q40

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1 Introduction

We investigate the initial value problem

$$\partial_t f + v \cdot \nabla_x f = Q(f) \ , \quad f(0,x,v) = f_0(x,v) \ , \tag{1}$$

where $f = f(t,x,v) \geq 0$ denotes a particle distribution function, depending on time $t \geq 0$, position $x \in \mathbb{T}^d$ (where $\mathbb{T}^d$ is a d-dimensional torus, i.e. a rectangular box with periodic boundary conditions), and velocity $v \in \mathbb{R}^d$.

The particles are fermions and the scattering operator Q (acting only in the v-direction) is a simple model for the interaction of the particles with a nonmoving

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background medium with constant temperature

$$Q(f) = \int_{\mathbb{R}^d} [M(1-f)f' - M'(1-f')f]\mathrm{d}v' \,. \qquad (2)$$

Here $f' = f(t,x,v')$ and $M(v) = (2\pi)^{-d/2}e^{-|v|^2/2}$ is the normalized Maxwellian. The factors $(1-f)$ and $(1-f')$ take into account the Pauli exclusion principle. The values of the distribution function have to respect the bounds $0 \le f \le 1$.

A kinetic equation with the same scattering operator, but also including acceleration of the particles by a given electric field, has been considered in [11]. Existence and uniqueness for initial value problems with $x \in \mathbb{R}^d$ (nonperiodic) have been proven and the macroscopic (diffusion) limit has been carried out.

A fermion Boltzmann equation modelling elastic particle-particle collisions has been studied by Dolbeault [7]. More elaborate models than (2) for the scattering of fermions due to a background medium or to a different species of particles have been considered in the modelling of charge transport in semiconductors ([9], [10], [3]). Existence and uniqueness and/or macroscopic limits are the subject of these studies.

Here we are interested in the long time behaviour of solutions of (1). It is characterized by two properties: conservation of total mass and entropy dissipation. The first is a consequence of the conservation property $\int_{\mathbb{R}^d} Q(f)\mathrm{d}v = 0$ of the scattering operator and of the periodic boundary conditions in position space:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v\mathrm{d}x = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} f_0(x,v)\mathrm{d}v\mathrm{d}x \,.$$

Before stating the entropy dissipation property, we write the collision operator in the form

$$Q(f) = \int_{\mathbb{R}^d} MM'(1-f)(1-f')(F'-F)\mathrm{d}v' \,,$$

with $F = \frac{f}{M(1-f)}$. Then, by the antisymmetry of the integrand with respect to v and v', it is easily shown that

$$\int_{\mathbb{R}^d} Q(f)\chi(F)\mathrm{d}v = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')(\chi(F)-\chi(F'))\mathrm{d}v'\mathrm{d}v \le 0 \,, \qquad (3)$$

for arbitrary increasing functions χ. As a consequence, if an entropy is defined by $H = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} S(f)\mathrm{d}v\mathrm{d}x$ with

$$\frac{\mathrm{d}S}{\mathrm{d}f} = \chi\left(\frac{f}{M(1-f)}\right) \,,$$

then the entropy dissipation equality

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(f)\chi(F)\mathrm{d}v\mathrm{d}x \qquad (4)$$

follows. If χ is chosen strictly increasing, then (by (3)) the entropy dissipation rate on the right hand side of (4) only vanishes if the (local) equilibrium condition $F = \kappa(t,x)$ is satisfied and the distribution function is the Fermi-Dirac distribution

$$f(t,x,v) = f_l(t,x,v) = \frac{\kappa(t,x)M(v)}{1+\kappa(t,x)M(v)}\,. \tag{5}$$

Note that in this statement, $\kappa(t,x)$ can be chosen arbitrarily. In the following, however, we shall denote by f_l defined by (5) the local equilibrium distribution associated to an arbitrary (nonequilibrium) distribution f, where κ is chosen by fixing the position density:

$$\int_{\mathbb{R}^d} f_l(t,x,v)\mathrm{d}v = \rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v\,. \tag{6}$$

Continuing our argument, we expect (because of the entropy dissipation equation) that for large times f approaches an equilibrium distribution, thus making the right hand side of the transport equation (1) vanish. The left hand side then only vanishes for constant κ. Thus, we expect f to converge to the global equilibrium

$$f_\infty(v) = \frac{\kappa_\infty M(v)}{1+\kappa_\infty M(v)}\,,$$

where κ_∞ is determined by mass conservation:

$$|\mathbb{T}^d|\int_{\mathbb{R}^d} f_\infty(v)\mathrm{d}v = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} f_0(x,v)\mathrm{d}v\mathrm{d}x\,.$$

From (4) a weak version of this statement can be proven (see, e.g., [4], [1]).

Recently, Desvillettes and Villani have developed a strategy for proving strong convergence to equilibrium for nonhomogeneous (in position) kinetic equations. It includes quantitative estimates on convergence rates. They have applied their approach to linear equations with a Fokker-Planck scattering operator and a confining potential [6] as well as, in a monumental work [5], to the Boltzmann equation of gas dynamics. Linear models have also been considered in [8] and [2].

In this work, the method of Desvillettes and Villani is applied to (1), (2). Its main point is to overcome the following difficulty: The right hand side of the entropy dissipation equation vanishes when the distribution is in local equilibrium. Thus, the entropy might stop decaying without f having reached the global equilibrium f_∞. As an input, the method requires certain bounds (uniform in time) on the distribution function and on its derivatives with respect to position. Whereas these could be proved for the linear problems in [6], [8], and [2], the have to be assumed for the Boltzmann equation. For the nonlinear problem (1), methods from [11] can be used to prove the propagation of bounds for the initial conditions in terms of Fermi-Dirac distributions. The boundedness of x-derivatives, however, has to be assumed.

In the following section, the boundedness result is proved, the method is outlined, and the main result is stated. The detailed (rather involved) computations and estimates are collected in section 3.

2 Preliminaries and main result

Theorem 1. *Assume there exist constants* $\kappa_-, \kappa_+ > 0$ *such that*

$$f_-(v) \le f_0(x,v) \le f_+(v)\,, \qquad \text{with } f_\pm(v) = \frac{\kappa_\pm M(v)}{1+\kappa_\pm M(v)}\,,$$

for all $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$. *Then there is a unique solution* $f(t,x,v)$ *of (1), (2) satisfying the same bounds:*

$$f_-(v) \le f(t,x,v) \le f_+(v)\,, \tag{7}$$

for all $t > 0$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$.

Proof. For details we refer to [11]. We only outline the proof of the bounds (7). The existence proof is based on a fixed point iteration on the set $\mathcal{V}$ of distribution functions satisfying (7). For $f \in \mathcal{V}$, we define the next iterate g by solving

$$\partial_t g + v\cdot\nabla_x g = M\int_{\mathbb{R}^d} f'\mathrm{d}v' - g\int_{\mathbb{R}^d}(Mf' + M'(1-f'))\mathrm{d}v'\,,$$

$$g(0,x,v) = f_0(x,v)\,.$$

The difference $r = g - f_-$ satisfies

$$r(0,x,v) = f_0(x,v) - f_-(v) \ge 0$$

and

$$\begin{aligned}
&\partial_t r + v\cdot\nabla_x r + r\int_{\mathbb{R}^d}(Mf' + M'(1-f'))\mathrm{d}v' \\
&= (1-f_-)M\int_{\mathbb{R}^d} f'\mathrm{d}v' - f_-\int_{\mathbb{R}^d} M'(1-f')\mathrm{d}v' \ge Q(f_-) = 0\,.
\end{aligned}$$

Nonnegativity of r and, thus, the lower bound $g \ge f_-$ follows. Analogously, $g \le f_+$ and, therefore, $g \in \mathcal{V}$ is shown. □

Note that this ensures that $(1-f)$ is bounded away from zero, which we will make use of frequently.

In the following, relative entropies will be used for measuring the distance between distributions. Some arbitrariness comes from the freedom to choose the function χ in (4). We define the relative entropy of f with respect to g by

$$H(f|g) := \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} S(f,g)\mathrm{d}v\mathrm{d}x\,,$$

with

$$S(f,g) = \int_g^f \ln\frac{z(1-g)}{g(1-z)}\mathrm{d}z = f\ln\frac{f(1-g)}{g(1-f)} + \ln\frac{1-f}{1-g}\,. \tag{8}$$

With this choice the relative entropy $H(f|f_\infty)$ coincides with the total entropy defined in the introduction for $\chi(z) = \ln(z)/\kappa_\infty$). Until now this choice seems somewhat artificial, but we will further comment on it after explaining the strategy to derive the convergence result. We shall need the derivatives

$$\frac{\partial S}{\partial f} = \ln \frac{f(1-g)}{g(1-f)}, \qquad \frac{\partial^2 S}{\partial f^2} = \frac{1}{f(1-f)}. \tag{9}$$

By $S(f,f) = \frac{\partial S}{\partial f}(f,f) = 0$ and $\frac{\partial^2 S}{\partial f^2} > 0$, the relative entropy has the desired property to measure the distance between f and g. Actually, a stronger statement is true:

Lemma 1. *Let f and g satisfy (7). Then there exist constants $c_1, c_2 > 0$, such that*

$$c_1 \|f-g\|_M^2 \le H(f|g) \le c_2 \|f-g\|_M^2$$

with the weighted L^2-norm

$$\|f\|_M^2 := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{f^2}{M} \, \mathrm{d}v \mathrm{d}x \,.$$

Proof. By (9) and the mean value theorem,

$$S(f,g) = \frac{(f-g)^2}{2\phi(1-\phi)},$$

with ϕ lying between f and g. In particular, ϕ also satisfies (7). As a consequence

$$\frac{M}{c_2} \le 2\phi(1-\phi) \le \frac{M}{c_1}$$

holds with appropriate constants c_1, c_2, completing the proof. □

A second important property is what we would call a nonlinear version of the Pythagorean theorem:

Lemma 2. *The relative entropy is additive with respect to the local equilibrium,*

$$H(f|f_l) + H(f_l|f_\infty) = H(f|f_\infty) \,.$$

Proof. A straightforward computation gives

$$H(f|f_l) + H(f_l|f_\infty) = H(f|f_\infty) + \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f - f_l) \mathrm{d}v \, \ln \frac{\kappa_\infty}{\kappa} \, \mathrm{d}x \,.$$

The integral with respect to velocity vanishes because of (6). □

Also since

$$\frac{\partial S}{\partial f}(f, f_\infty) = \ln \frac{f}{\kappa_\infty M(1-f)},$$

we can use (4), (3) with $\chi(z) = \ln(z/\kappa_\infty)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) = -\frac{1}{2} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F') \ln \frac{F}{F'} \,\mathrm{d}v'\mathrm{d}v\mathrm{d}x\,. \qquad (10)$$

As mentioned above, the right hand side vanishes when f is a Fermi-Dirac distribution (such that F is independent of v). The basic idea of the Desvillettes-Villani approach is to show that in such a situation the entropy dissipation cannot remain zero as long as $f \neq f_\infty$. This is done by estimating the entropy dissipation in terms of the relative entropy of f with respect to the local equilibrium f_l and by deriving a second order differential inequality for $H(f|f_l)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) &\leq -c_3 H(f|f_l)\,, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|f_l) &\geq c_4 H(f|f_\infty) - c_5 H(f|f_l)^{1-1/n}\,. \end{aligned} \qquad (11)$$

It is the main contribution of this work to prove that these inequalities hold for appropriate $c_3, c_4, c_5 > 0$ and a positive integer n. This will be done in the following section. The proof requires the estimates from Theorem 1 and additional smoothness assumptions on the solution.

A result from [6] for systems of differential inequalities of the form (11) can then be used to get the desired convergence theorem:

Theorem 2. *Let the assumptions of Theorem 1 hold and let the solution f of (1) satisfy*

$$\left\| \frac{\partial^{k_1+\ldots+k_d} f}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}(t,\cdot,\cdot) \right\|_M \leq c_6\,, \qquad \forall\, k_1 + \ldots + k_d \leq n \quad \text{and} \quad \forall\, t > 0\,, \qquad (12)$$

for a constant c_6 and a positive integer n. Then there exists a constant $c_7 > 0$ such that

$$H(f|f_\infty) \leq c_7 t^{1-n}\,.$$

Remark 1: For the Fokker-Planck collision operator, a smoothness result like (12) was proven in [6] even for nonsmooth initial conditions by exploiting a hypoellipticity property. Here, one could only hope for propagation of regularity as in [8], assuming smoothness of the initial data.

Remark 2: In principle we have the freedom to choose in (8) any entropy of the type

$$S_\chi(f,g) = \int_g^f \chi\left(\frac{z(1-g)}{g(1-z)}\right) \mathrm{d}z\,,$$

with an arbitrary monotone increasing function χ. Since the biggest difficulty is to deduce the second inequality in the system (11) we choose the relative entropy such that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_l)$ becomes as simple as possible. The calculation for the general case leads to the choice $\chi = \ln(\cdot/\kappa_\infty)$, which corresponds to (8).

Corollary 1. *With the assumptions of the previous theorem, there exists $c_8 > 0$, such that*

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} \leq c_8 t^{(1-n)/2} .$$

Proof. The Cauchy-Schwarz inequality implies

$$\|g\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \frac{|g|}{\sqrt{M}}\sqrt{M}\,\mathrm{d}v\mathrm{d}x \leq \sqrt{|\mathbb{T}^d|}\,\|g\|_M .$$

The result now follows from Lemma 1 and from Theorem 2. □

3 Derivation of Differential Inequalities

Throughout this section c will be a positive real constant that may change from line to line.

Lemma 3. *Let the assumptions of Theorem 2 hold. Then there is a constant $c_3 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) \leq -c_3 H(f|f_l) .$$

Proof. We have to estimate the entropy production (10). Note that by the mean value theorem

$$\ln\frac{F}{F'} = \frac{F-F'}{\Phi}$$

holds, with Φ between F and F'. Also, by (7), we have $\kappa_- \leq \Phi \leq \kappa_+$. This gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) &\leq -\frac{1}{2\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')^2\mathrm{d}v'\mathrm{d}v\mathrm{d}x \\
&\leq -\frac{1}{2\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(1-f_l)(1-f_l') \\
&\qquad\qquad \times(F-F_l-(F'-F_l'))^2\mathrm{d}v'\mathrm{d}v\mathrm{d}x ,
\end{aligned}$$

where we have used $F_l = \frac{f_l}{M(1-f_l)} = \kappa = F_l'$. Expanding the square and using

$$F - F_l = \frac{f-f_l}{M(1-f)(1-f_l)} , \tag{13}$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) \;\le\; -\frac{1}{\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M(1-f)(1-f_l)\mathrm{d}v \int_{\mathbb{R}^d} \frac{(f-f_l)^2}{M(1-f)(1-f_l)}\mathrm{d}v\mathrm{d}x$$
$$+\frac{1}{\kappa_+}\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}^d}(f-f_l)\mathrm{d}v\right)^2\mathrm{d}x\,.$$

The last term vanishes by the requirement (6) on the local equilibrium. From (7),

$$0 < 1-f_+(0) \le 1-f, 1-f_l \le 1$$

follows and, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) \le -c\|f-f_l\|_M^2\,.$$

An application of Lemma 1 completes the proof. □

Now we shall prove the second inequality in (11). A straightforward computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_l) = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\left(\partial_t f\ln\frac{F}{F_l} - \partial_t f_l\frac{f-f_l}{f_l(1-f_l)}\right)\mathrm{d}v\mathrm{d}x$$
$$= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\left(-v\cdot\nabla_x f\ln F + v\cdot\nabla_x f\ln\kappa + Q(f)\ln F - \frac{\partial_t\kappa}{\kappa}(f-f_l)\right)\mathrm{d}v\mathrm{d}x\,.$$

The first term on the right hand side vanishes by the divergence theorem (with respect to the x-variable) and the last one by (6), leaving

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_l) = \int_{\mathbb{T}^d}\nabla_x\cdot J\,\ln\kappa\,\mathrm{d}x + \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}Q(f)\ln F\,\mathrm{d}v\mathrm{d}x = A + B\,, \tag{14}$$

with the flux density $J = \int_{\mathbb{R}^d} vf\,\mathrm{d}v$ (which vanishes for $f = f_l$).

For the computation of the time derivative of A we need the momentum balance equation

$$\partial_t J + \nabla_x\cdot P = \int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,,$$

where we shall split the pressure tensor into a local equilibrium part and a remainder:

$$P = \int_{\mathbb{R}^d} v\otimes vf\,\mathrm{d}v = \int_{\mathbb{R}^d} v\otimes vf_l\,\mathrm{d}v + \int_{\mathbb{R}^d} v\otimes v(f-f_l)\,\mathrm{d}v = P_l + \tilde{P}\,.$$

Differentiating (6) with respect to time and the continuity equation $\partial_t\rho + \nabla_x\cdot J = 0$ lead to

$$\frac{\partial_t\kappa}{\kappa} = \frac{-\nabla_x\cdot J}{\int_{\mathbb{R}^d} f_l(1-f_l)\mathrm{d}v}\,.$$

With these preparations we obtain

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot P_l)\mathrm{d}x + \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot \tilde{P})\mathrm{d}x \\
&\quad - \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot \int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{T}^d} \frac{(\nabla_x \cdot J)^2}{\int_{\mathbb{R}^d} f_l(1-f_l)\mathrm{d}v}\,\mathrm{d}x\,. \qquad (15)
\end{aligned}
$$

Note that for $f = f_l$ all terms on the right hand side except the first vanish. This term is responsible for moving f out of local equilibrium as long as it is not in global equilibrium. For estimating it we need

$$
\nabla_x \cdot P_l = \int_{\mathbb{R}^d} v \otimes v\, f_l(1-f_l)\mathrm{d}v\, \frac{\nabla_x \kappa}{\kappa}\,. \qquad (16)
$$

The integral is an isotropic tensor which is positive definite since f_l satisfies (7):

$$
\int_{\mathbb{R}^d} v \otimes v\, f_l(1-f_l)\mathrm{d}v \geq \mathrm{Id} \int_{\mathbb{R}^d} v_i^2 f_-(1-f_+)\mathrm{d}v\,. \qquad (17)
$$

The first term on the right hand side of (15) can, thus, be estimated by

$$
\int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot P_l)\mathrm{d}x \geq c\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)}\,.
$$

Now we estimate the remaining three terms in (15) one by one. First:

$$
\begin{aligned}
\left| \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot \tilde{P})\mathrm{d}x \right| &\leq c \int_{\mathbb{T}^d} |\nabla_x \kappa| \int_{\mathbb{R}^d} |v|^2 |\nabla_x (f - f_l)| \mathrm{d}v\,\mathrm{d}x \\
&\leq c \int_{\mathbb{T}^d} |\nabla_x \kappa| \sqrt{\int_{\mathbb{R}^d} |v|^4 M\,\mathrm{d}v \int_{\mathbb{R}^d} \frac{|\nabla_x (f - f_l)|^2}{M} \mathrm{d}v}\;\mathrm{d}x \\
&\leq c\|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|\nabla_x (f - f_l)\|_M\,.
\end{aligned}
$$

Second (similarly):

$$
\left| \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot \int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,\mathrm{d}x \right| \leq c\|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|Q(f)\|_M\,.
$$

Now we need to quantify the behaviour of $Q(f)$ near f_l.

Lemma 4. *Q is a bounded operator and moreover*

$$
\|Q(f)\|_M \leq c\|f - f_l\|_M\,.
$$

Proof.

$$\|Q(f)\|_M^2 \le c \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M \left(\int_{\mathbb{R}^d} M'|F-F'|\mathrm{d}v'\right)^2 \mathrm{d}v\,\mathrm{d}x\,.$$

The integral in parentheses can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^d} M'|F-F'|\mathrm{d}v' &\le |F-F_l| + \int_{\mathbb{R}^d} M'|F'-F_l'|\mathrm{d}v' \\ &\le |F-F_l| + \sqrt{\int_{\mathbb{R}^d} M(F-F_l)^2\mathrm{d}v}\,.\end{aligned}$$

Estimating the square of the sum by the sum of the squares we get

$$\|Q(f)\|_M^2 \le c \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M(F-F_l)^2 \mathrm{d}v\,\mathrm{d}x\,.$$

Now the result of the lemma follows from (13) and the boundedness of $(1-f)$ away from zero. □

Second, continued:

$$\left|\int_{\mathbb{T}^d} \frac{\nabla_x\kappa}{\kappa}\cdot\int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,\mathrm{d}x\right| \le c\|\nabla_x\kappa\|_{L^2(\mathbb{T}^d)}\|f-f_l\|_M\,.$$

Third (last term in (15)):

$$\begin{aligned}&\int_{\mathbb{T}^d} \frac{(\nabla_x\cdot J)^2}{\int_{\mathbb{R}^d} f_l(1-f_l)\mathrm{d}v}\,\mathrm{d}x \le c\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}^d} v\cdot\nabla_x(f-f_l)\mathrm{d}v\right)^2\mathrm{d}x \\ &\le c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}|v|^2M\mathrm{d}v\int_{\mathbb{R}^d}\frac{|\nabla_x(f-f_l)|^2}{M}\mathrm{d}v\,\mathrm{d}x = c\|\nabla_x(f-f_l)\|^2\,.\end{aligned}$$

Collecting our results so far, we have proved

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} \ge c\|\nabla_x\kappa\|_{L^2(\mathbb{T}^d)}^2 \;&-\; \tilde c\big(\|\nabla_x\kappa\|_{L^2(\mathbb{T}^d)}\|\nabla_x(f-f_l)\|_M \\ &+\|\nabla_x\kappa\|_{L^2(\mathbb{T}^d)}\|f-f_l\|_M + \|\nabla_x(f-f_l)\|_M^2\big)\,,\end{aligned}$$

implying

$$\frac{\mathrm{d}A}{\mathrm{d}t} \ge c\|\nabla_x\kappa\|_{L^2(\mathbb{T}^d)}^2 - \tilde c\big(\|f-f_l\|_M^2 + \|\nabla_x(f-f_l)\|_M^2\big)\,. \tag{18}$$

The $\nabla_x\kappa$ term drives the solution out of local equilibria because it remains non zero as long as κ is different from the constant κ_∞. A Poincaré type estimate will help us to describe this by means of relative entropy.

Lemma 5. $\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq cH(f_l|f_\infty)$ *with* $c > 0$.

Proof. Since $\frac{\mathrm{d}\rho}{\mathrm{d}\kappa} = \frac{1}{\kappa}\int_{\mathbb{R}^d} f_l(1-f_l)\mathrm{d}v$ is bounded from above and away from zero (by (7)),

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\|\nabla_x \rho\|^2_{L^2(\mathbb{T}^d)}$$

with $c > 0$. Introducing $\rho_\infty = \int_{\mathbb{R}^d} f_\infty \mathrm{d}v$ and noting that $\int_{\mathbb{T}^d}(\rho - \rho_\infty)\mathrm{d}x = 0$, a Poincaré estimate gives

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\|\rho - \rho_\infty\|^2_{L^2(\mathbb{T}^d)}\mathrm{d}x\,,$$

with a possibly different, but still positive constant c. On the other hand,

$$|\rho - \rho_\infty| \geq c|\kappa - \kappa_\infty| = c\frac{|f_l - f_\infty|}{M(1-f_l)(1-f_\infty)} \geq c\frac{|f_l - f_\infty|}{M}\,,$$

where the first inequality follows again from the boundedness of $\frac{\mathrm{d}\rho}{\mathrm{d}\kappa}$ and the last from (7). This implies

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M(\kappa - \kappa_\infty)^2 \mathrm{d}v\mathrm{d}x \geq c\|f_l - f_\infty\|^2_M\,.$$

An application of Lemma 1 completes the proof. □

It remains to estimate the time derivative of the second term in (14). Using (3), this term can be written as

$$B = -\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')\ln\frac{F}{F'}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x\,.$$

The computation of the time derivative is facilitated by the fact that the integrand is symmetric with respect to f and f':

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} \quad &= \quad \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'\frac{1-f'}{1-f}\Bigg[M(1-f)(F-F')\ln\frac{F}{F'} \\ &\qquad\qquad - \ln\frac{F}{F'} - (F-F')\frac{1}{F}\Bigg]\partial_t f\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x\,.\end{aligned}$$

The term multiplying $\partial_t f$ in the integrand can by estimated (using (7)) by $cM'|F-F'|$. As a consequence,

$$\left|\frac{\mathrm{d}B}{\mathrm{d}t}\right| \leq c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'|F-F'|(|v|\,|\nabla_x f| + |Q(f)|)\mathrm{d}v'\mathrm{d}v\mathrm{d}x$$

holds. With $|F-F'| \le |f-f_l|/M + |f'-f_l'|/M'$, the right hand side is bounded by the sum of four terms, which we estimate one by one. First:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M' \frac{|f-f_l|\,|Q(f)|}{M}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \|f-f_l\|_M \|Q(f)\|_M\,.$$

Second:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f'-f_l'|\,|Q(f)|\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \|f-f_l\|_M \|Q(f)\|_M\,.$$

Third:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f'-f_l'|\,|v|\,|\nabla_x f|\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \int_{\mathbb{R}^d} |v|^2 M\,\mathrm{d}v\,\|f-f_l\|_M \|\nabla_x(f-f_\infty)\|_M\,.$$

The fourth term is the most difficult to estimate. Here we have to make a small concession on the exponent. We use $|f-f_l| \le cM$:

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'|v|\frac{|f-f_l|\,|\nabla_x f|}{M}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\left|\frac{f-f_l}{\sqrt{M}}\right|^{1-\varepsilon}\frac{|\nabla_x f|}{\sqrt{M}}|v|M^{\varepsilon/2}\mathrm{d}v\mathrm{d}x \\
&\le c\left(\int_{\mathbb{R}^d}|v|^{2/\varepsilon}M\,\mathrm{d}v\right)^{\varepsilon/2}\|f-f_l\|_M^{1-\varepsilon}\|\nabla_x(f-f_\infty)\|_M\,.
\end{aligned}$$

Since the Maxwellian has finite moments of arbitrary order, ε can be made arbitrarily small. Collecting the four estimates, using Lemma 4 and the fact that $\|f-f_l\|_M$ is bounded, we have

$$\left|\frac{\mathrm{d}B}{\mathrm{d}t}\right| \le c\left(\|f-f_l\|_M^2 + \|\nabla_x(f-f_\infty)\|_M\|f-f_l\|_M^{1-\varepsilon}\right)\,.$$

This implies together with (14), (18), and Lemma 5

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|f_l) \ge cH(f_l|f_\infty) \quad - \quad & \tilde{c}\big(\|\nabla_x(f-f_l)\|_M^2 + \|f-f_l\|_M^2 \\
& + \|\nabla_x(f-f_\infty)\|_M\|f-f_l\|_M^{1-\varepsilon}\big)\,. \qquad (19)
\end{aligned}$$

The next step is the derivation of bounds for the norms of the gradients. The interpolation inequality

$$\|\nabla_x u\|_{L^2(\mathbb{T}^d)} \le c\|u\|_{L^2(\mathbb{T}^d)}^{1-1/n}\|u\|_{H^n(\mathbb{T}^d)}^{1/n}$$

and the Hölder inequality imply

$$\begin{aligned}
\|\nabla_x g\|_M^2 = \int_{\mathbb{R}^d}\frac{1}{M}\|\nabla_x g\|_{L^2(\mathbb{T}^d)}^2\mathrm{d}v \quad &\le \quad c\int_{\mathbb{R}^d}\left(\frac{1}{M}\|g\|_{L^2(\mathbb{T}^d)}^2\right)^{1-1/n}\left(\frac{1}{M}\|g\|_{H^n(\mathbb{T}^d)}^2\right)^{1/n}\mathrm{d}v \\
&\le \quad c\|g\|_M^{2(1-1/n)}\left(\int_{\mathbb{R}^d}\frac{1}{M}\|g\|_{H^n(\mathbb{T}^d)}^2\mathrm{d}v\right)^{1/n}\,.
\end{aligned}$$

By assumption (12) of Theorem 2 the last factor is bounded uniformly in time for $g = f - f_l$ and $g = f - f_\infty$. This gives

$$\|\nabla_x(f - f_l)\|_M^2 \le c\|f - f_l\|_M^{2(1-1/n)}$$

and, with the Young inequality,

$$\begin{aligned} \|\nabla_x(f - f_\infty)\|_M \|f - f_l\|_M^{1-\varepsilon} &\le c\|f - f_\infty\|_M^{1-1/n}\|f - f_l\|_M^{1-\varepsilon} \\ &\le \delta\|f - f_\infty\|_M^2 + c_\delta\|f - f_l\|_M^{2(1-\varepsilon)n/(n+1)} \end{aligned}$$

Now we choose $\varepsilon = n^{-2}$ (such that $(1-\varepsilon)n/(n+1) = 1 - 1/n$) and use the above inequalities and the Lemmata 1 and 2 in (19), to obtain the desired result

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|f_l) \ge c_4 H(f|f_\infty) - c_5 H(f|f_l)^{1-1/n} \,.$$

This completes the derivation of the differential inequalities (11) and, thus, the proof of Theorem 2.

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