Existence and convergence to equilibrium of a kinetic model for cometary flows

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Abstract. A kinetic equation with a relaxation time model for wave-particle collisions is considered. Similarly to the BGK-model of gas dynamics, it involves a projection onto the set of equilibrium distributions, nonlinearly dependent on moments of the distribution function. An earlier existence result is extended to bounded domains with reflecting boundaries and to initial conditions permitting vacuum regions. The long time behaviour is investigated. Convergence on compact time intervals (shifted to infinity) to the set of equilibrium solutions is proven. The set of smooth equilibrium solutions is computed.

Key words: kinetic equation, wave-particle collision operator, cometary flows, convergence to equilibrium

AMS subject classification: 41A60, 35Q20, 76P05

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1 Introduction

We investigate initial-boundary value problems for the kinetic equation

$$(1) \qquad \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q_{\mathbf{u}_f}(f) = P_{\mathbf{u}_f}(f) - f\,,$$

where $f = f(t,\mathbf{x},\mathbf{v}) \geq 0$ denotes a particle distribution function, depending on time $t > 0$, position $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, Ω is a bounded domain with piecewise C^1 boundary), and velocity $\mathbf{v} \in \mathbb{R}^d$. The collision operator $Q_{\mathbf{u}_f}$ is used in quasilinear plasma theory as a simplified model for wave-particle interaction in cometary flows. The map $P_{\mathbf{u}}$ is a projection onto the set of distribution functions isotropic around the mean velocity $\mathbf{u}$:

$$P_{\mathbf{u}}(f)(\mathbf{v}) = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(\mathbf{u} + |\mathbf{v}-\mathbf{u}|\boldsymbol{\omega})\,d\boldsymbol{\omega}\,,$$

with S^{d-1} and $|S^{d-1}|$ denoting the unit sphere in $\mathbb{R}^d$ and its $(d-1)$-dimensional Lebesgue measure, respectively. By $\mathbf{u}_f$ we denote the mean velocity of the distribution function f, i.e. the ratio of the macroscopic momentum density $\mathbf{m}_f$ and the mass density ρ_f:

$$\mathbf{u}_f = \frac{\mathbf{m}_f}{\rho_f}\,, \qquad \rho_f = \int_{\mathbb{R}^d} f\,d\mathbf{v}\,, \qquad \mathbf{m}_f = \int_{\mathbb{R}^d} \mathbf{v} f\,d\mathbf{v}\,.$$

The kinetic equation (1) is considered subject to initial conditions

$$(2) \qquad f(0,\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v})\,,$$

for $(\mathbf{x},\mathbf{v}) \in \Omega \times \mathbb{R}^d$. For the initial data we shall use the following assumptions:

$$(3) \qquad \exists\, p > 1 : \quad f_0 \in L^p(\Omega\times\mathbb{R}^d)\,, \quad f_0 \geq 0\,,$$

$$(4) \qquad \exists\, r > 1 : \quad (1+|\mathbf{v}|^r) f_0 \in L^1(\Omega\times\mathbb{R}^d)\,.$$

We impose reflecting boundary conditions

$$(5) \qquad f(t,\mathbf{x},\mathbf{v}) = f(t,\mathbf{x},\mathbf{v}')\,,$$

for $t > 0$, $\mathbf{x} \in \partial\Omega$, with specular or reverse reflection, i.e.,

$$(6) \qquad \text{a) } \mathbf{v}' = \mathbf{v} - 2(\mathbf{n}(\mathbf{x})\cdot\mathbf{v})\mathbf{n}(\mathbf{x})\,, \qquad \text{or b) } \mathbf{v}' = -\mathbf{v}\,,$$

where $\mathbf{n}(\mathbf{x})$ denotes a unit normal along $\partial\Omega$.

For the physics modelled by (1), we refer the reader to [11], [18], [19] and [20]. Let us only remark that the collision operator describes the scattering of cosmic rays (energetic particles) in an astrophysical plasma, caused by random irregularities in the ambient field [11]. The collision operator considered here is a simplified relaxation time model, comparable to the BGK model of gas dynamics [16], [17]. We are treating a dimensionless version where the relaxation time has been chosen as reference time.

A mathematical treatment of (1) has been started in [5] and [6], where macroscopic limits have been computed formally. In [7], the whole space problem ($\Omega = \mathbb{R}^d$) is considered and existence of weak solutions of the initial value problem is proven. Also, macroscopic conservation laws, an entropy dissipation equality, and the propagation of higher order moments is shown. For the initial data, several strong assumptions are used in [7], such as boundedness, existence of moments up to the second order and, most importantly, a positivity assumption guaranteeing that vacuum is avoided.

In section 3 of the present work, an existence theorem under milder assumptions is proven. In particular, the occurrence of vacuum is allowed. Also the result is stronger compared to that in [7] in the sense that weak solutions are shown to be mild solutions. Although we restrict our attention to bounded position domains with reflective boundaries, our results can be easily carried over to the whole space problem. In section 3 we also prove results on the propagation of moments and on the validity of macroscopic conservation laws, formally derived in section 2.

Our results for the long time behaviour of solutions of (1)–(5) correspond to those of Desvillettes [8] for gas dynamics. In particular, in section 4 we prove that, on a compact time interval shifted to infinity, the solution of (1)–(5) converges to a solution of (1), (5), lying in the null set of the collision operator. This result is complemented in section 5 by the computation of all smooth equilibrium solutions of (1) and by the identification of the subset satisfying the boundary conditions (5).

We conclude the introduction by mentioning questions this work leaves open. The set of solutions of (1), (5), constructed in section 5, is infinite dimensional, a fact which relies on the nature of the collisions, for which in the homogeneous case all isotropic functions are solutions. On the other hand, only a finite number of conserved quantities is given in section 3. Therefore, the large time limit cannot be determined uniquely from the initial data. This also inhibits attempts to obtain stronger convergence results including the rate of convergence by an entropy-entropy dissipation approach for nonhomogeneous kinetic equations, recently developped by Desvillettes and Villani and carried out for different linear models [9], [12] and for the Boltzmann equation [10]. For future work we leave the idea to overcome the underdetermination by imposing thermalizing boundary conditions forcing the solution to a given global equilibrium function (cf. [3], [1] for the

Boltzmann equation).

2 Preliminaries

First, we collect some formal properties of the collision operator (see, e.g., [7]).

Lemma 1 *Let* $\mathbf{u} \in \mathbb{R}^d$, $f, g \in C_0^\infty(\mathbb{R}^d)$, $f, g \geq 0$, $\rho_f > 0$, $\varphi \in C^\infty([0,\infty))$. *Then*

(i) Symmetry: $\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f) g \, d\mathbf{v} = \int_{\mathbb{R}^d} f Q_{\mathbf{u}}(g) \, d\mathbf{v}$.

(ii) Collision invariants: $\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f) \varphi(|\mathbf{v} - \mathbf{u}|) \, d\mathbf{v} = 0$.

(iii) Equilibrium: $Q_{\mathbf{u}_f}(f) = 0$, *if and only if there exist* $\mathbf{u} \in \mathbb{R}^d$ *and* $F \in C_0^\infty([0,\infty))$, *such that* $f(\mathbf{v}) = F(|\mathbf{v} - \mathbf{u}|^2/2)$.

(iv) H-theorem: For monotonically increasing χ,

$$\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f) \chi(f) \, d\mathbf{v} = -\int_{\mathbb{R}^d} \big[f - P_{\mathbf{u}}(f)\big] \big[\chi(f) - \chi(P_{\mathbf{u}}(f))\big] \, d\mathbf{v} \leq 0 \,.$$

Since the collision invariants of the form $\varphi(|\mathbf{v} - \mathbf{u}_f|)$ depend nonlocally on the distribution function they do not lead to conservation laws. The only f-independent collision invariants of $Q_{\mathbf{u}_f}$ are linear combinations of 1, the components of $\mathbf{v}$, and $|\mathbf{v}|^2 = |\mathbf{v} - \mathbf{u}_f|^2 - |\mathbf{u}_f|^2 + 2\mathbf{u}_f \cdot \mathbf{v}$. Local conservation laws (for mass ρ_f, momentum $\mathbf{m}_f$, and energy $E_f = \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} f \, d\mathbf{v}$) are only produced by these:

$$\partial_t \begin{pmatrix} \rho_f \\ \mathbf{m}_f \\ E_f \end{pmatrix} + \nabla_{\mathbf{x}} \cdot \int_{R^d} \begin{pmatrix} \mathbf{v} \\ \mathbf{v} \otimes \mathbf{v} \\ \mathbf{v}|\mathbf{v}|^2/2 \end{pmatrix} f \, d\mathbf{v} = 0 \,. \tag{7}$$

Note that, by the symmetry of the momentum flux tensor, we also have conservation of the $d(d-1)/2$ components of angular momentum:

$$\partial_t \int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f \, d\mathbf{v} + \nabla_{\mathbf{x}} \cdot \int_{R^d} \mathbf{v}(x_i v_j - x_j v_i) f \, d\mathbf{v} = 0 \,, \tag{8}$$

$1 \leq i < j \leq d$. For the determination of globally conserved quantities in (1)–(5), we have to consider the effect of the reflexive boundary. The boundary conditions (5) conserve mass and energy such that these quantities are globally conserved:

$$\int_\Omega \begin{pmatrix} \rho_f \\ E_f \end{pmatrix} d\mathbf{x} = \int_\Omega \begin{pmatrix} \rho_{f_0} \\ E_{f_0} \end{pmatrix} d\mathbf{x}\,. \tag{9}$$

For reverse reflexive boundaries (6) b) no other conserved quantities are known. In the case of specular reflection (6) a) the component of angular momentum corresponding to the index pair (i,j) is globally conserved, if Ω has the corresponding rotational symmetry, i.e., with $(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_d) \in \Omega$, all points $(x_1,\ldots,\sqrt{x_i^2+x_j^2}\cos\varphi,\ldots,\sqrt{x_i^2+x_j^2}\sin\varphi,\ldots,x_d)$, $\varphi \in \mathbb{R}$, also belong to Ω. Then, for the flux of angular momentum through the boundary $\partial\Omega$ we have

$$\int_{\partial\Omega}\int_{\mathbb{R}^d} (\mathbf{n}\cdot\mathbf{v})(x_i v_j - x_j v_i) f \, d\mathbf{v} d\sigma = 2\int_{\partial\Omega} (x_i n_j - x_j n_i) \int_{\mathbb{R}^d} (\mathbf{n}\cdot\mathbf{v})^2 f \, d\mathbf{v} d\sigma = 0\,,$$

since $x_i n_j - x_j n_i = 0$ in the rotationally symmetric case. Consequently,

$$\int_\Omega\int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f \, d\mathbf{v} d\mathbf{x} = \int_\Omega\int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f_0 \, d\mathbf{v} d\mathbf{x}\,. \tag{10}$$

More generally, every $(d-2)$-dimensional affine space in $\mathbb{R}^d$ can serve as 'rotation axis' instead of the subspace $\{x_i = x_j = 0\}$. Summarizing, the number of globally conserved quantities in (1)–(5) is between 2 and $2 + d(d-1)/2$ (the latter, when Ω is a ball with specularly reflecting boundary).

The H-theorem Lemma 1 (iv) with $\chi = pf^{p-1}$, $p > 1$, leads to the entropy dissipation equality

$$\partial_t \int_\Omega\int_{\mathbb{R}^d} f^p \, d\mathbf{v} d\mathbf{x} = -p \int_\Omega\int_{\mathbb{R}^d} \left[f - P_{\mathbf{u}_f}(f)\right]\left[f^{p-1} - P_{\mathbf{u}}(f)^{p-1}\right] d\mathbf{v} d\mathbf{x}\,, \tag{11}$$

playing a central role in our study of the convergence to equilibrium below.

In the existence analysis, we shall use continuity properties of the collision operator (derived in [7]):

Lemma 2 *Let $\tau > 0$, $1 \le p, q \le \infty$, $f \in L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))$, $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$ in $L^1((0,\tau)\times\Omega)^d$. Then*

(i) $\lim_{n\to\infty} P_{\mathbf{u}_n}(f) = P_{\mathbf{u}}(f)$ *in* $L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))$ *for* $p, q < \infty$,

(ii) $\|P_{\mathbf{u}}(f)\|_{L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))} \le \|f\|_{L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))}$.

In the existence analysis below, the semigroup generated by the free streaming operator subject to the reflecting boundary conditions is used. The solution operator $T(t)$ for

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0\,,$$

subject to (2), (5), i.e., $f(t,\mathbf{x},\mathbf{v}) = (T(t)f_0)(\mathbf{x},\mathbf{v})$, is a strongly continuous positivity preserving contraction semigroup on $L^p(\Omega\times\mathbb{R}^d)$ for $1\le p<\infty$ (see [4], section 9.3). A mild formulation of (1)–(5) is then given by the Duhamel formula

$$f(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}_f}(f)(s)\,ds\,. \tag{12}$$

Note that solutions of (12) are also weak solutions on finite time intervals $(0,\tau)$ in the sense that [2]

$$\begin{aligned} &\int_D f(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\,d\mathbf{v}d\mathbf{x}dt + \int_\Omega\int_{\mathbb{R}^d} f_0\varphi(t=0)\,d\mathbf{v}d\mathbf{x} \\ &= \int_D fQ_{\mathbf{u}_f}(\varphi)\,d\mathbf{v}d\mathbf{x}dt\,, \end{aligned} \tag{13}$$

for all $\varphi\in\mathcal{D}_\tau = \{\varphi\in C_0^\infty([0,\tau)\times\overline{\Omega}\times\mathbb{R}^d):\ \varphi \text{ satisfies } (5)\}$.

3 Existence and conservation laws

The existence proof follows the approach of [7] extended by a final step where solutions with vacuum regions are allowed.

As a first step we solve a linearized problem.

Lemma 3 *Let (3) hold and* $\mathbf{u}\in L^\infty((0,\infty)\times\Omega)$ *be given. Then*

$$f(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}}(f)(s)\,ds \tag{14}$$

has a unique nonnegative solution $f\in C([0,\infty);\ L^p(\Omega\times\mathbb{R}^d))$ *satisfying*

$$\|f(t)\|_{L^p(\Omega\times\mathbb{R}^d)} \le \|f_0\|_{L^p(\Omega\times\mathbb{R}^d)}\,. \tag{15}$$

Proof: Existence and uniqueness are the consequence of a standard contraction argument. The estimate (15) follows from the contractivity of $T(t)$, Lemma 2 (ii), and an application of the Gronwall lemma. Continuity in t is a straightforward consequence of (14) and of Lemma 2. ∎

The next result is concerned with the propagation of moments in the linear problem.

Lemma 4 *Let the assumptions of Lemma 3 and (4) hold. Then, for the solution of (14),* $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega\times\mathbb{R}^d))$ *holds.*

Proof: In the weak formulation

$$
\begin{aligned}
(16)\qquad & \int_D f(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\,d\mathbf{v}d\mathbf{x}dt + \int_\Omega\int_{\mathbb{R}^d} f_0\varphi(t=0)\,d\mathbf{v}d\mathbf{x} \\
&= \int_D Q_{\mathbf{u}}(f)\varphi\,d\mathbf{v}d\mathbf{x}dt\,,
\end{aligned}
$$

of (14) we choose as a test function $\varphi(t,\mathbf{x},\mathbf{v}) = \theta(t)\Phi(|\mathbf{v}|^2/V)(1+|\mathbf{v}|^r) \in \mathcal{D}_\tau$ with $\theta \in C_0^\infty([0,\tau))$, $\Phi \in C_0^\infty([0,\infty))$, $\Phi(y) = 1$ for $y < 1$, $V > 0$. Using $(a+b)^r \le c_r(a^r + b^r)$, we estimate

$$
\begin{aligned}
(17)\qquad & \int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)P_{\mathbf{u}}(f)\,d\mathbf{v} \le c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}|^r + |\mathbf{u}|^r)P_{\mathbf{u}}(f)\,d\mathbf{v} \\
&= c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}|^r+|\mathbf{u}|^r)f\,d\mathbf{v} \le C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}|^r)f\,d\mathbf{v}\,,
\end{aligned}
$$

where $C_r = c_r^2 + c_r$ is a constant depending only on r. Since u is bounded, we estimate further

$$
(18)\qquad C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}|^r)f\,d\mathbf{v} \le c\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v}
$$

to show that the limit of (16) as $V\to\infty$ implies a differential inequality of the form

$$
\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v}d\mathbf{x} \le c\int_\Omega\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v}d\mathbf{x}\,.
$$

The proof of the lemma is now completed by an application of the Gronwall lemma. ∎

Our first result for the nonlinear problem (1) assumes vacuum avoiding initial data (like in [7]).

Theorem 1 *Let (3), (4) hold. Moreover, assume* $f_0(\mathbf{x},\mathbf{v}) \ge g(|\mathbf{v}|)$ *with* g *having strictly positive density* $\rho_g \ge \gamma > 0$*. Then (1) has a mild, global, nonnegative solution* $f \in C([0,\infty);\ L^p(\Omega\times\mathbb{R}^d))$ *satisfying (15),* $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega\times\mathbb{R}^d))$*, and* $\mathbf{u}_f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega))$*.*

Proof: The proof follows along the lines of the proof of Theorem 1 in [7] and is only outlined here.

As in [7] and [16], we introduce a velocity truncation

$$\varphi_n(\mathbf{u}) = \begin{cases} \mathbf{u} & \text{for } |\mathbf{u}| \le n, \\ n\frac{\mathbf{u}}{|\mathbf{u}|} & \text{for } |\mathbf{u}| > n. \end{cases}$$

Then the solvability of the regularized problem

$$f_n(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}_n}(f_n)(s)\,ds\,, \tag{19}$$

with $\mathbf{u}_n = \varphi_n(\mathbf{u}_{f_n})$ is shown by a fixed point iteration on the mean velocity, where the fixed point operator involves solving linear problems as in Lemma 3. Lemma 4 is used for the control of moments providing compactness of the mean velocity by an averaging lemma and by the boundedness from below of the mass density.

For passing to the limit $n \to \infty$, we need uniform bounds on moments of f_n. We proceed as in the proof of Lemma 4. Only the estimate (18) with u_n and f_n instead of u and f needs to be redone without using boundedness of the velocity. Since, obviously, $|\mathbf{u}_n| \le |\mathbf{u}_{f_n}|$ holds, we estimate

$$C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r + |\mathbf{u}_n|^r)f_n\,d\mathbf{v} \le c\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f_n\,d\mathbf{v}\,,$$

where we have used

$$\rho_f|\mathbf{u}_f|^r \le \int_{\mathbb{R}^d}|\mathbf{v}|^r f\,d\mathbf{v}\,,$$

an application of Jensen's inequality (for the convex function $v \to |v|^r$ with the measure f/ρ_f) which applies here and avoids the use of more elaborated controls like Perthame and Pulvirenti in [17] for the BGK-model. The convergence in $L^1((0,\tau)\times\Omega)$ of $\mathbf{u}_n = \varphi_n(\mathbf{u}_{f_n})$ to $\mathbf{u}_f$ (where f is the weak limit of f_n) is shown as in [7]. The limit $n \to \infty$ in the weak version of (19) (compare (13)) can now be carried out (applying Lemma 2), and the proof is complete. ■

In the next step, we remove the assumption of positive densities. For this purpose we have to extend the definition of the collision operator in a trivial way: Let $f, vf \in L^1(\mathbb{R}^d)$, $f \ge 0$. Then $Q(f) := P(f) - f$ with

$$P(f) := \begin{cases} P_{\mathbf{u}_f}(f) & \text{for } \rho_f > 0\,, \\ 0 & \text{for } \rho_f = 0\,. \end{cases}$$

Note that Lemma 2 (ii) remains true for P (instead of $P_{\mathbf{u}}$). Also the statements of Lemma 1 obviously remain true for Q (instead of $Q_{\mathbf{u}}$ and $Q_{\mathbf{u}_f}$).

Theorem 2 *Let (3), (4) hold. Then (1) (with $Q_{\mathbf{u}_f}$ replaced by Q) has a mild, global, nonnegative solution $f \in C([0,\infty);\ L^p(\Omega \times \mathbb{R}^d))$ satisfying (15) and $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega \times \mathbb{R}^d))$.*

Proof: For $n \in \mathbb{N}$, the modified initial data

$$f_{0n}(\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v}) + \frac{1}{n} e^{-|\mathbf{v}|^2}$$

satisfy the assumptions of Theorem 1 guaranteeing the existence of a weak solution f_n of (12) (with f_0 replaced by f_{0n}). Note that f_{0n} satisfies (3) and (4) uniformly in n.

By (15), f_n is bounded in $L^\infty((0,\infty);\ L^p(\Omega \times \mathbb{R}^d))$ uniformly in n and, thus, a subsequence converges weakly to a limit f. Compactness of the moments up to order 1 is deduced as above, such that, for a further subsequence, we have

$$\rho_{f_n} \to \rho_f\,, \quad \mathbf{m}_{f_n} \to \mathbf{m}_f \qquad \text{in} \quad L^1(G)\,,$$

with $G = (0,\tau) \times \Omega$. By the Egoroff theorem, for a further subsequence, ρ_{f_n} converges almost uniformly on G, i.e., for every $\varepsilon > 0$ there exists $N_\varepsilon \subset G$ with $|N_\varepsilon| \le \varepsilon$ such that ρ_{f_n} converges to ρ_f uniformly on $G \setminus N_\varepsilon$. Now $G \setminus N_\varepsilon$ is decomposed further into subsets A_ε and B_ε, where $\rho_f \le \varepsilon$ and $\rho_f > \varepsilon$ holds, respectively. For a test function $\varphi \in \mathcal{D}_\tau$, the integral

$$\int_{G \times \mathbb{R}^d} (P(f_n) - P(f))\varphi \, dt\, d\mathbf{x}\, d\mathbf{v} \tag{20}$$

is also split into three contributions according to the decomposition $G = A_\varepsilon \cup B_\varepsilon \cup N_\varepsilon$. For the first part we have the estimate

$$\left| \int_{A_\varepsilon \times \mathbb{R}^d} (P(f_n) - P(f))\varphi \, dt\, d\mathbf{x}\, d\mathbf{v} \right| \le (2\varepsilon + a_n)\tau |\Omega| \sup |\varphi|\,,$$

with $a_n \to 0$ as $n \to \infty$, by Lemma 2 (ii) and the uniform convergence of the density. In B_ε, $\mathbf{u}_n = \mathbf{m}_{f_n}/\rho_{f_n}$ is well defined for n large enough and converges to $\mathbf{u}_f$ in $L^1(B_\varepsilon)$. By the symmetry property Lemma 1 (i), the second contribution to (20) can be written as

$$\int_{B_\varepsilon \times \mathbb{R}^d} (f_n P_{\mathbf{u}_n}(\varphi) - f P_{\mathbf{u}_f}(\varphi))\, dt\, d\mathbf{x}\, d\mathbf{v}$$

which converges to zero for $n \to \infty$ by the weak convergence of f_n, the strong convergence of $\mathbf{u}_n$, and by Lemma 2 (i). Finally, the third contribution to (20) is estimated by

$$\left| \int_{N_\varepsilon \times \mathbb{R}^d} (P(f_n) - P(f))\varphi \, dt\, d\mathbf{x}\, d\mathbf{v} \right| \le \sup |\varphi| \int_{N_\varepsilon} (\rho_{f_n} + \rho_f)\, dt\, d\mathbf{x}\,.$$

The right hand side and, thus, (20) tend to zero for $n \to \infty$, $\varepsilon \to 0$ by the convergence of ρ_{f_n} to ρ_f and by $|N_\varepsilon| \le \varepsilon$. This proves that we can pass to the limit in the weak formulation of the problem for f_n. ■

In the last result of this section the formal computations of section 2 concerning entropy dissipation and conservation laws are justified. Proofs are analogous to those of the corresponding results of [7] and omitted here. We only note that for local conservation of energy we have to assume the existence of the energy flux vector initially, since a dispersion result used in [7] for the whole space problem does not apply here.

Theorem 3 *Let the assumptions of Theorem 2 hold. Then:*

(i) The entropy dissipation equation (11) holds.

(ii) Let $r \ge 2$ in (4). Then the global conservation laws (9), (10) hold (the latter in case of specular reflection and rotational symmetry).

(iii) Let $r \ge 3$ in (4). Then the local conservation laws (7), (8) hold in the distributional sense.

4 Convergence to equilibrium

Theorem 4 *Let the assumptions of Theorem 2 hold with $r \ge 2$ in (4). Then, for every sequence $t_n \to \infty$, there exists a subsequence (again denoted by t_n) and a function $f_\infty \in L^\infty(\mathbb{R}; L^p(\Omega \times \mathbb{R}^d))$, such that, for every $T > 0$,*

$$f_n(t, \mathbf{x}, \mathbf{v}) := f(t_n + t, \mathbf{x}, \mathbf{v}) \to f_\infty(t, \mathbf{x}, \mathbf{v}) \quad \textit{weakly in } L^p((-T,T) \times \Omega \times \mathbb{R}^d)\,,$$
$$Q(f_\infty) = 0\,,$$
$$\textstyle\int_{\mathbb{R}\times\Omega\times\mathbb{R}^d} f_\infty(\partial_t \varphi + \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi)\, d\mathbf{v} d\mathbf{x} dt = 0\,,$$

for every $\varphi \in C_0^\infty((0,T) \times \overline{\Omega} \times \mathbb{R}^d)$ satisfying the boundary conditions (5), i.e., f_∞ is an equilibrium solution of the free streaming equation satisfying the reflection boundary conditions.

Proof: We first prove the result with a subsequence possibly depending on T. Then the statement of the theorem follows from a diagonal procedure.

The weak convergence of f_n (up to a subsequence) to a limit f_∞ follows from the boundedness of $\|f(t,\cdot,\cdot)\|_{L^p(\Omega\times\mathbb{R}^d)}$ uniformly in time. By Theorem 3 (ii), ρ_{f_n} and E_{f_n} are bounded in $L^1((-T,T)\times\Omega)$ uniformly in n. The same holds for $\mathbf{m}_{f_n}$ by the interpolation

$$|\mathbf{m}_f| \le \sqrt{\rho_f E_f} \le \frac{\rho_f + E_f}{2}\,.$$

For passing to the limit $n \to \infty$ we proceed exactly as in Theorem 2 proving that

$$\partial_t f_\infty + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\infty = Q(f_\infty)$$

and the boundary conditions hold in the weak sense indicated in the formulation of the theorem. To prove f_∞ to be an equilibrium distribution, we apply the entropy dissipation equation (11) (valid by Theorem 3 (i)). As a first step we note that the assumptions (3), (4) imply the validity of assumption (3) also for every q between 1 and p by interpolation. Let us pick, in particular, $q = \min\{p, 2\}$. The entropy dissipation equation (11) then implies

$$\int_0^\infty \int_\Omega \int_{\mathbb{R}^d} \left[f - P(f)\right]\left[f^{q-1} - P(f)^{q-1}\right] d\mathbf{v}\, d\mathbf{x}\, dt < \infty\,,$$

and, hence,

$$\int_{-T}^{T} \int_\Omega \int_{\mathbb{R}^d} \left[f_n - P(f_n)\right]\left[f_n^{q-1} - P(f_n)^{q-1}\right] d\mathbf{v}\, d\mathbf{x}\, dt$$

converges to zero. The convexity and definiteness of the function $C(x, y) = (x - y)(x^{q-1} - y^{q-1})$ for $1 < q \le 2$ and the weak convergence of f_n and $P(f_n)$ allow to pass to the limit and conclude

$$f_\infty = P(f_\infty)\,,$$

completing the proof. ■

5 Smooth equilibrium solutions

In this section we compute all smooth solutions of the system

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0\,, \qquad Q(f) = 0\,. \tag{21}$$

In a second step, we shall identify those solutions satisfying the reflecting boundary conditions (5). We remark that we have to leave open the gap between the above weak convergence result and the consideration of smooth solutions.

We distinguish between subsets of $(t, \mathbf{x})$-space where the density ρ_f (and, thus, the distribution function f) vanishes and where ρ_f is positive. The following arguments holds for a connected component $D \subset \mathbb{R}^{d+1}$ of $\{\rho_f > 0\}$.

Then, by $Q(f) = 0$, there exists a mean velocity $\mathbf{u}(t, \mathbf{x})$ and a function $F(t, \mathbf{x}, \xi)$ such that

$$f(t, \mathbf{x}, \mathbf{v}) = F\left(t, \mathbf{x}, \frac{|\mathbf{v} - \mathbf{u}(t, \mathbf{x})|^2}{2}\right).$$

Substituting this representation into the free streaming equation and introducing the change of variables $\mathbf{v} \mapsto (\xi, \boldsymbol{\omega}) \in [0, \infty) \times S^{d-1}$, defined by $\mathbf{v} = \mathbf{u} + \boldsymbol{\omega}\sqrt{2\xi}$, gives

$$-(2\xi\partial_\xi F)\boldsymbol{\omega}^{tr}(\nabla_{\mathbf{x}}\mathbf{u})\boldsymbol{\omega} + \sqrt{2\xi}\,\boldsymbol{\omega}\cdot(\nabla_{\mathbf{x}}F - \partial_\xi F D_t\mathbf{u}) + D_t F = 0\,, \tag{22}$$

where $\boldsymbol{\omega}^{tr}$ is the transpose of $\boldsymbol{\omega}$ and the material derivative is denoted by $D_t = \partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}$. We exploit this equation using the following simple linear algebra lemma.

Lemma 5 *Let* $\mathrm{A} \in \mathbb{R}^{d\times d}$ *and* $\mathbf{b} \in \mathbb{R}^d$. *Then*

$$\boldsymbol{\omega}^{tr}\mathrm{A}\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\mathbf{b} = 0 \qquad \textit{for all } \boldsymbol{\omega} \in S^{d-1} \tag{23}$$

holds iff $\mathbf{b} = 0$ *and* A *is skew-symmetric.*

Proof: The matrix A can be replaced by its even part $\tilde{\mathrm{A}} = \frac{1}{2}(\mathrm{A} + \mathrm{A}^{tr})$ in (23). The odd part can be arbitrary. By a rotation of $\boldsymbol{\omega}$, $\tilde{\mathrm{A}}$ can be assumed as diagonal. The choices $\boldsymbol{\omega} = \pm\mathbf{e}_j$, $j = 1, \ldots, d$ prove $\tilde{\mathrm{A}} = \mathbf{b} = 0$. ■

Keeping $(t, \mathbf{x}, \xi)$ fixed and varying $\boldsymbol{\omega} \in S^{d-1}$ in (22), we deduce that

$$\nabla_{\mathbf{x}}F = \partial_\xi F D_t\mathbf{u} \tag{24}$$

holds, and that the matrix

$$\mathrm{A} = D_t F\,\mathrm{Id} - 2\xi\partial_\xi F\nabla_{\mathbf{x}}\mathbf{u}$$

is skew-symmetric. Application of the curl to (24) leads to

$$0 = (\partial_\xi\partial_{x_i}F D_t u_j - \partial_\xi\partial_{x_j}F D_t u_i) + \partial_\xi F(\partial_{x_i}D_t u_j - \partial_{x_j}D_t u_i)\,,$$

for $1 \le i < j \le d$. Using (24) again for the computation of the components of $\nabla_{\mathbf{x}}F$ shows that the first term vanishes. Also the existence and positivity of the macroscopic density in D implies that for every $(t, \mathbf{x}) \in D$ there exists $\xi > 0$ such that $\partial_\xi F(t, \mathbf{x}, \xi) \ne 0$. Thus, the curl of $D_t\mathbf{u}$ vanishes in D. This implies that locally in D a potential $\tilde{g}(t, \mathbf{x})$ exists such that

$$D_t\mathbf{u} = \nabla_{\mathbf{x}}\tilde{g}$$

holds. Now (24) implies the existence of a function $F_0(t,z)$ such that

$$F(t,\mathbf{x},\xi) = F_0(t,\xi + \tilde{g}(t,\mathbf{x}))\,. \tag{25}$$

The fact that the diagonal elements of A vanish, imply that the diagonal elements of $\nabla_{\mathbf{x}}\mathbf{u}$ are identical, $\partial_{x_i}u_i(t,\mathbf{x}) = \sigma(t,\mathbf{x})$, and

$$\mathrm{A}_{ii} = \partial_t F_0 + \partial_z F_0 (D_t\tilde{g} + 2\sigma\tilde{g} - 2z\sigma) = 0\,. \tag{26}$$

Similarly to above, we argue that for every t in the projection of D onto the t-axis there exists a z-interval of positive length, such that $\partial_z F_0(t,z) \neq 0$. This implies that the coefficients $D_t\tilde{g} + 2\sigma\tilde{g}$ and σ in (26) are independent of $\mathbf{x}$. By (26), F_0 can be written in the form $F_0(t,z) = \psi(\alpha(t)z + \beta(t))$ and, therefore, for F we have $F(t,\mathbf{x},\xi) = \psi(\alpha(t)\xi + g(t,\mathbf{x}))$ (with $g = \alpha\tilde{g} + \beta$). Returning to the equation $\mathrm{A}_{ii} = 0$, we deduce

$$\dot{\alpha} = 2\sigma\alpha\,, \qquad \dot{(\cdot)} = \tfrac{d\cdot}{dt}\,, \tag{27}$$

$$D_t g = 0\,. \tag{28}$$

More information on the form of $\alpha(t)$ and $g(t,\mathbf{x})$ is derived similarly to Desvillettes [8]. The gradient of the vector field $\mathbf{u}(t,\mathbf{x}) - \sigma(t)\mathbf{x}$ is skew-symmetric. From Lemma 1 in [8] we conclude that $\mathbf{u}$ can be written in the form

$$\mathbf{u}(t,\mathbf{x}) = \sigma(t)\mathbf{x} + \Lambda(t)\mathbf{x} + \mathbf{C}(t)$$

with skew-symmetric $\Lambda(t)$. Inserting this representation in

$$\nabla_{\mathbf{x}} g = \alpha D_t \mathbf{u}\,, \tag{29}$$

(which is a consequence of (24)) leads to the necessary condition $\dot{\Lambda} + 2\sigma\Lambda = 0$ and, thus, $\Lambda(t) = \Lambda_0/\alpha(t)$, for the right hand side of (29) to be a gradient (Λ_0 is an arbitrary constant skew-symmetric matrix). Now (29) is integrated with respect to $\mathbf{x}$ and the result is substituted in (28). The left hand side of (28) becomes a quadratic polynomial in $\mathbf{x}$. Equating coefficients to zero gives a system of ordinary differential equations for the unknown t-dependent quantities, which can be solved explicitly. We omit the details of the computation and only state the result: The distribution function can be written in the form

$$f(t,\mathbf{x},\mathbf{v}) = \psi\left(\alpha(t)\frac{|\mathbf{v} - \mathbf{u}(t,\mathbf{x})|^2}{2} + g(t,\mathbf{x})\right)\,. \tag{30}$$

There exist three constant scalars $a, b, c \in \mathbb{R}$, two constant vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^d$, and a constant skew-symmetric matrix $\Lambda_0 \in \mathbb{R}^{d\times d}$ such that

$$
\begin{aligned}
\alpha(t) &= at^2 + 2bt + c\,, \\
g(t,\mathbf{x}) &= \frac{ac-b^2}{\alpha(t)}\frac{|\mathbf{x}|^2}{2} - \frac{|\Lambda_0\mathbf{x}|^2}{2\alpha(t)} \\
&\quad + \left(\mathbf{A} + \frac{1}{\alpha(t)}(\Lambda_0 - at - b)(\mathbf{A}t+\mathbf{B})\right)\cdot\mathbf{x} - \frac{|\mathbf{A}t+\mathbf{B}|^2}{2\alpha(t)}\,, \\
\mathbf{u}(t,\mathbf{x}) &= \frac{1}{\alpha(t)}((at+b)\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{A}t + \mathbf{B})\,.
\end{aligned}
\tag{31}
$$

So far this holds only locally in D. We intend to make the result global. Let us consider the intersection of the domains of two local representations of the form (30), (31). We shall prove that the function ψ and the constants have to be the same in both representations. However, an obvious source of nonuniqueness has to be eliminated first. We require $\alpha(t) > 0$ and a normalization of the coefficients of $\alpha(t)$: $a^2 + b^2 + c^2 = 1$. This can be achieved by a rescaling of the argument of ψ. At zeroes of $\alpha(t)$ velocity moments of the distribution (30) are not defined, hence the restriction to time intervals where $\alpha(t)$ does not change sign.

The mean velocity $\mathbf{u}(t,\mathbf{x})$ has to be the same in both representations and, thus, also the diagonal elements $\dot\alpha/(2\alpha)$ of its gradient. With the normalization condition this implies that $\alpha(t)$ and therefore also the coefficients a, b, and c are the same. Now it is an easy consequence of the formula for $\mathbf{u}$ in (31) that also the other coefficients Λ_0, $\mathbf{A}$, and $\mathbf{B}$ are the same. Finally, we conclude that the functions ψ in both representations have to be identical.

Summarizing, we have proven the following.

Theorem 5 *Let f be a smooth solution of (21), and let D be an open connected subset of $\mathbb{R}^{d+1}$ where ρ_f is positive. Then, for $(t,\mathbf{x}) \in D$, $\mathbf{v} \in \mathbb{R}^d$, $f(t,\mathbf{x},\mathbf{v})$ can be written in the form (30), (31).*

Finally, we study the effect of reflecting boundary conditions. In the work by Desvillettes [8] on the Boltzmann and BGK equations it is shown that for Maxwellian equilibria (i.e., $\psi(y) = e^{-y}$) solving the free streaming problem with reflexive boundaries, vacuum cannot occur locally. The proof can easily be extended to any strictly positive ψ. It essentially relies on the fact that particles are spread with arbitrary velocities. In the more general situation discussed here, where ψ may have compact support, vacuum regions can be part of an equilibrium distribution. For a classification of solutions of the form (30), (31), and the interplay between vacuum and non-vacuum regions we refer to [13]. Here we only discuss a special situation needed below, namely non-vacuum islands within a vacuum region. These are defined by inequalities of the form

$$g(t, \mathbf{x}) \leq g_0 \,, \tag{32}$$

assuming $\psi(g) = 0$ for $g \geq g_0$. The inequality (32) defines a bounded region in $\mathbf{x}$-space iff the matrix $(ac - b^2)\mathrm{Id} + \Lambda_0^2$ is positive definite. In [13] it is shown that the form (30), (31), is invariant under Galilean transformations $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0 + \mathbf{v}_0 t$, $\mathbf{v} \to \mathbf{v} + \mathbf{v}_0$, and that the regularity of $(ac - b^2)\mathrm{Id} + \Lambda_0^2$ is sufficient for eliminating the vectors $\mathbf{A}$ and $\mathbf{B}$ by an appropriate choice of $\mathbf{x}_0$ and $\mathbf{v}_0$. Thus, after the Galilean transformation, (32) can be written as

$$\mathbf{x}^{tr}((ac - b^2)\mathrm{Id} + \Lambda_0^2)\mathbf{x} \leq 2 g_0 \alpha(t) \,.$$

Therefore, the nonvacuum region is an ellipsoid growing with time, whose center moves with constant velocity. The significance of this result for the following is that non-vacuum islands cannot be contained in bounded domains for large times.

First, we consider the case of reverse reflection boundary conditions.

Theorem 6 *Let $f(t, \mathbf{x}, \mathbf{v})$ be a smooth solution of (21) for $(t, \mathbf{x}, \mathbf{v}) \in (0, \infty) \times \Omega \times \mathbb{R}^d$, satisfying the boundary conditions (5), (6)b). Then*

$$f(t, \mathbf{x}, \mathbf{v}) = \psi\left(|\mathbf{v}|^2\right) \,, \tag{33}$$

for an appropriate function ψ.

Proof: Assume first that there exists a time $t_0 > 0$ and a position $\mathbf{x}_0 \in \partial\Omega$ such that $\rho_f(t_0, \mathbf{x}_0) > 0$. Then ρ_f is positive in a neighbourhood $D \subset (0, \infty) \times \overline{\Omega}$ of $(t_0, \mathbf{x}_0)$. By Theorem 5, f is of the form (30), (31) in D, and the boundary condition implies that the mean velocity vanishes along the boundary:

$$(at + b)\mathbf{x} + \Lambda_0 \mathbf{x} + \mathbf{A}t + \mathbf{B} = 0 \,, \qquad (t, \mathbf{x}) \in D \,, \quad \mathbf{x} \in \partial\Omega \,.$$

By varying t, $a\mathbf{x} + \mathbf{A} = b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B} = 0$ holds for $\mathbf{x} \in \partial D \cap \partial\Omega$. Varying $\mathbf{x}$ in the first equation implies $a = \mathbf{A} = 0$. On the other hand, it is easily shown that the rank of a matrix of the form $b\mathrm{Id} + \Lambda_0$ with skew-symmetric Λ_0 cannot be 1. Therefore, if b or Λ_0 do not vanish, the system $b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B} = 0$ defines a set of codimension bigger than 1, which cannot contain the hypersurface $\partial D \cap \partial\Omega$. This implies $b = \Lambda_0 = \mathbf{B} = 0$, proving that f is of the desired form (33) in D and, by smoothness, $D = (0, \infty) \times \overline{\Omega}$.

It remains to consider the situation when $\rho_f = 0$ on $(0,\infty) \times \partial\Omega$. Then, if the total mass is positive, Ω must contain a non-vacuum island. However, by the above arguments, this situation cannot persist as $t \to \infty$. The non-vacuum region eventually has to contain part of the boundary, putting us back into the case considered previously. ■

Theorem 6 is a complete result for the reverse reflection case. For specular reflection, the situation is more complicated.

Theorem 7 *Let $f(t,\mathbf{x},\mathbf{v})$ be a smooth solution of (21) for $(t,\mathbf{x},\mathbf{v}) \in (0,\infty)\times\Omega\times\mathbb{R}^d$, satisfying the boundary conditions (5), (6)a). Suppose the boundary of the $\mathbf{x}$-component $\tilde{\Omega}(t) = \{\mathbf{x} \in \Omega :\ (t,\mathbf{x}) \in D\}$ of a nonvacuum region D (as above) contains the boundary of Ω: $\partial\Omega \subset \partial\tilde{\Omega}(t)$, for $t_1 < t < t_2$. Then*

$$f(t,\mathbf{x},\mathbf{v}) = \psi\left(|\mathbf{v}|^2 + \mathbf{v}^{tr}\Lambda_0(\mathbf{x}-\mathbf{x}_0)\right), \tag{34}$$

for an appropriate function ψ, a skew symmetric matrix Λ_0, and a point $\mathbf{x}_0$, such that Ω is invariant under rotations of the form $\mathbf{x} \mapsto \Pi(\mathbf{x}) + e^{\Lambda_0 s}\mathbf{x}$, $s \in \mathbb{R}$, where Π denotes the orthogonal projection onto the rotation axis defined by $\Lambda_0(\mathbf{x}-\mathbf{x}_0) = 0$. If Ω does not have any rotational symmetry, $f(t,\mathbf{x},\mathbf{v}) = \psi(|\mathbf{v}|^2)$ holds.

Proof: For specular reflection boundary conditions the mean velocity satisfies $\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}(x) = 0$ along the boundary, implying $(a\mathbf{x}+\mathbf{A})\cdot\mathbf{n}(x) = 0$ and $(b\mathbf{x}+\Lambda_0\mathbf{x}+\mathbf{B})\cdot\mathbf{n}(x) = 0$ for $\mathbf{x} \in \partial\Omega$. Therefore solutions of the ODEs

$$\frac{d\mathbf{x}}{ds} = a\mathbf{x} + \mathbf{A}\,, \qquad \frac{d\mathbf{x}}{ds} = b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B} \tag{35}$$

with initial data on $\partial\Omega$ remain on $\partial\Omega$, which is a bounded set. However, solutions of (35) only remain bounded iff $a = b = \mathbf{A} = 0$ and $\mathbf{B} \in \mathrm{rg}\Lambda_0$. The solutions of the second ODE then describe rotations around an axis determined by $\Lambda_0(\mathbf{x}-\mathbf{x}_0) = 0$, where we have set $\mathbf{B} = -\Lambda_0\mathbf{x}_0$.

So far we have shown that $\tilde{\Omega}(t)$ is independent of t and that the representation (34) is valid for $(t,\mathbf{x}) \in (0,\infty)\times\tilde{\Omega}$. We cannot exclude the possibility, however, that Ω contains a vacuum region away from the boundary. By the arguments already used previously, there cannot exist a non-vacuum island within this vacuum region and, thus, (34) holds for all $\mathbf{x} \in \Omega$. ■

Remark 1 *Taking a closer look at the situation discussed at the end of the proof, we note that a vacuum region in the interior of the domain would be defined by an inequality of the form $|\Lambda_0(\mathbf{x}-\mathbf{x}_0)| \le \mathrm{const}$. Since the rank of a skew symmetric matrix is even, this inequality can only describe a bounded region, iff the space dimension d is even.*

The question arises if the result of the preceding theorem remains true even without the assumption of non-vacuum along the whole boundary. We conjecture that the answer is positive for $d = 3$ and prove the corresponding result for the case when Ω is a ball. Finally, an example will show that the answer is negative for $d > 3$.

Theorem 8 *Let Ω be a ball in $\mathbb{R}^3$ with center $\mathbf{x}_0$, and let $f(t, \mathbf{x}, \mathbf{v})$ be a smooth solution of (21) for $(t, \mathbf{x}, \mathbf{v}) \in (0, \infty) \times \Omega \times \mathbb{R}^3$, satisfying the boundary conditions (5), (6)a). Then f can be written as (34) with an appropriate function ψ and a skew-symmetric matrix Λ_0.*

Proof: As in the proof of Theorem 6, it suffices to consider the case where a non-vacuum region D extends to the boundary. As in the proof of Theorem 7, solutions of (35) with initial data on $\partial D \cap \partial\Omega$ remain on the boundary. When a or $\mathbf{A}$ are different from zero, the solution trajectories of the first equation are straight lines which cannot be part of a sphere. Therefore, $a = \mathbf{A} = 0$.

For solutions of the second ODE, the requirement $(\mathbf{x} - \mathbf{x}_0) \cdot d\mathbf{x}/ds = 0$ for remaining on the sphere leads to

$$bR^2 + (b\mathbf{x}_0 + \Lambda_0\mathbf{x}_0 + \mathbf{B}) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

for all $\mathbf{x} \in \partial D \cap \partial\Omega$, where R denotes the radius of the sphere. However, for $b\mathbf{x}_0 + \Lambda_0\mathbf{x}_0 + \mathbf{B} \neq 0$ this defines a hyperplane, which cannot contain a nontrivial part of a sphere, implying $b = 0$ and $\mathbf{B} = -\Lambda_0\mathbf{x}_0$. Thus, the solution is of the form (34) in D and the $\mathbf{x}$-component of D is the intersection of the ball Ω with the exterior of a cylinder. In three-dimensional space there can only be one non-vacuum region of this form. Interior non-vacuum islands can be excluded like above and the proof is complete. ■

Remark 2 *The proof can be extended to more general domains whose boundaries do not contain solution trajectories of the ODEs (35).*

Finally, we present a solution for Ω being the ball with center at the origin and radius R in $\mathbb{R}^4$ with specularly reflecting boundary, which is not of the form (34): Let $0 < \rho_1, \rho_2 < R$ and $\rho_1^2 + \rho_2^2 \geq R^2$. Let Ω_1 be the subdomain of Ω defined by $x_1^2 + x_2^2 > \rho_1^2$, and Ω_2 be accordingly defined by $x_3^2 + x_4^2 > \rho_2^2$. Then Ω_1 and Ω_2 are nonempty and do not intersect. Equilibrium solutions of the form (34) with $\mathbf{x}_0 = 0$, the skew symmetric matrices

$$\Lambda_1 = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{pmatrix},$$

and $\mathbf{x}$-supports in Ω_1 and Ω_2, respectively, can be defined by choosing the appropriate ψ-functions with compact support. Because of $\Omega_1 \cap \Omega_2 = \{\}$ the sum of these solutions is also an equilibrium solution, but not of the form (34).

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