Burgers-Poisson: A nonlinear dispersive model equation

Klemens Fellner,[1] Christian Schmeiser[1]

Abstract. A dispersive model equation is considerd, which has been proposed by Whitham as a shallow water model, and which can also be seen as a caricature of two species Euler-Poisson problems. A number of formal properties as well as similarities to other dispersive equations is derived. A travelling wave analysis and some numerical tests are carried out. The equation features wave breaking in finite time. A local existence result for smooth solutions and a global existence result for weak entropy solutions is proved. Finally a small dispersion limit is carried out for situations where the solution of the limiting equation is smooth.

Key words: dispersive models, entropy solutions, small dispersion limit

AMS subject classification: 35L65, 35Q20

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[1]Institut für Angewandte und Numerische Mathematik, TU Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria.

1 Introduction and Motivation

In this paper, a nonlinear dispersive model problem is proposed, the Burgers-Poisson (BP)-system:

$$u_t + uu_x = \varphi_x, \tag{1}$$

$$\varphi_{xx} = \varphi + u, \tag{2}$$

where u and φ depend on $(t,x) \in (0,\infty) \times \mathbb{R}$, and subscripts denote partial derivatives. The Burgers equation (1) is driven by the right hand side φ_x, which is determined by solving the Poisson equation (2).

Using the Green's function $G(x) = -\frac{1}{2}e^{-|x|}$ (of $\partial_x^2 - 1$) to define the convolution operator

$$\varphi[u](x) = (G * u)(x) = \int_{\mathbb{R}} G(x-y)u(y)\,dy \tag{3}$$

the BP-system reduces to the single BP-equation

$$u_t + uu_x = \varphi_x[u], \tag{4}$$

with the obvious notation $\varphi_x[u] := (\varphi[u])_x$.

A rescaled version of (4) was considered by Whitham [27, Chapter 13.14] as a shallow water equation featuring weaker dispersivity than the Korteweg-de Vries (KdV)-equation.

The study of (1), (2) has been motivated by earlier work [10], [11] on two-species-Euler-Poisson (2SEP)-systems modelling the dynamics of 2 oppositely charged species of particles subject to Coulomb interaction. A simple version in dimensionless form is given by

$$\rho_t + (\rho u)_x = 0, \tag{5}$$

$$u_t + uu_x + \tfrac{\rho_x}{\rho} = \varphi_x, \tag{6}$$

$$\varepsilon\varphi_{xx} = \rho - e^{-\varphi}. \tag{7}$$

Here, the unknowns ρ, u and φ depend on position $x \in \mathbb{R}$ and time $t > 0$. The system (5), (6) are the isothermal Euler equations for the first species of particles with density ρ and velocity u. In the Poisson equation (7) for the electrostatic potential φ, the density of the second species is modelled by the equilibrium approximation $e^{-\varphi}$, resulting from an equation like (6) with the first two terms neglected and the opposite sign on the right hand side. The dimensionless parameter ε denotes the scaled Debye length.

In view of the formal similarities between the BP- and the 2SEP-system, we shall use the terms position, time, velocity and potential for the variables

x, t, u and φ, respectively. Note that the velocity (instead of the density) appears on the right hand side of the Poisson equation. Nevertheless one can think the BP-system as a caricature of the 2SEP-system with Burgers equation replacing the Euler equations, and the potential terms in (2) coming from a linearization of the two-species Poisson equation (7).

The BP-system has a number of interesting formal properties, collected in section 2. In particular, we mention its relation to the Camassa-Holm equation [4], [5] and the Benjamin-Ono equation [2], [24], its Galilean invariance, its Hamiltonian structure as well as the existence of an entropy.

In section 3, a general travelling wave analysis of the BP-system is performed recovering the result of Fornberg and Whitham [16] in the particular case of solitary waves. It turns out that the travelling wave structure of the BP-system and several versions of the 2SEP-system (see [10], [11]) are qualitatively equivalent. The section is completed with some numerical experiments.

In section 4, existence and uniqueness of smooth solutions locally in time for smooth initial data are proven. Recently, for two related problems, the Euler-Poisson system [13] and the Camassa-Holm equation [9], [25], global existence of smooth solutions has been shown under certain conditions on the initial data. The methods employed there do not apply to the BP-system. Also our numerical experiments with the BP-system indicate that comparable results are not true. A global existence result for weak entropy solutions with initial data in $BV(\mathbb{R})$ is also derived. A comparable result has recently been shown for a radiating gas model [21], which is obtained from the BP-system replacing u by $-u_x$ in the right hand side of (2).

Finally, the rescaling $x \to x/\varepsilon, t \to t/\varepsilon, 0 < \varepsilon \ll 1$, is introduced in (1), (2), leading to

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon, \tag{8}$$

$$\varepsilon^2 \varphi_{xx}^\varepsilon = \varphi^\varepsilon + u^\varepsilon. \tag{9}$$

A Chapman-Enskog expansion of (8), (9) recovers the Burgers equation with flux $(u^\varepsilon + 1)^2/2$ and the leading order perturbation $\varepsilon^2 u_{xxx}^\varepsilon$. For a rescaled system, there exists a direct asymptotic towards the KdV-equation for $\varepsilon \to 0$. The travelling wave analysis and numerical simulations suggest that the quasineutral limit $\varepsilon \to 0$ in general is a weak limit, both for the 2SEP- and the BP-systems. Here, a result is shown for the BP-system which has been proved for a 2SEP-system in [12] and for the radiating gas model in [21]: convergence to smooth solutions of the formal limit problem. In general this is only a local-in-time result since the limiting inviscid Burgers equation can develop singularities in finite time. The situation is the same for 2SEPsystem, but not for the radiating gas model, where the limiting equation is the viscous Burgers equation with global smooth solutions.

2 Formal Properties

Firstly, we rewrite the BP-system as a single differential equation for u. By appling $1 - \partial_x^2$ to (1) and using (2) on the resulting right hand side, we calculate:

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{10}$$

The terms in (10) correspond to those in the Camassa-Holm equation [4]:

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{11}$$

where the constant $\kappa \geq 0$ is related to the critical shallow water wave speed. Vice versa, the Camassa-Holm equation (11) can be written in "BP form":

$$u_t + uu_x = \varphi_x\,, \tag{12}$$

$$\varphi_{xx} = \varphi + 2\kappa u + u^2 + \tfrac{1}{2}u_x^2\,. \tag{13}$$

Note that for $\kappa = 1/2$ the BP-system is recovered by neglecting the two quadratic terms in (13).

The Camassa-Holm equation was introduced by Fokas and Fuchssteiner [14] as formally integrable bi-Hamiltonian generalization of the KdV-equation. Camassa and Holm [4] rediscovered it as shallow water equation by approximating the Hamiltonian for the vertically averaged incompressible Euler equations. By the bi-Hamiltonian property, they derived an infinite sequence of conservation laws and showed that the associated flows of this hierarchy are isospectral and, thus, completely integrable.

Most commonly (cf. [5], [9], [8], [25]), the Camassa-Holm equation (11) is considered with $\kappa = 0$. Then the Camassa-Holm equation possesses peaked soliton solutions (called peakons), attractive travelling waves of the form $u(x,t) = c\exp(-|x - ct|)$ and other breaking phenomena, which is desirable for a shallow water equation and in contrast to the KdV-behaviour. For some initial data (e.g. with sufficiently large negative slope [8], [25]) the solution develops verticality within finite time. On the other hand, global well-posedness was proved ([9], [25]) for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ provided that $\int |u_0|\,dx < \infty$ and $(1 - \partial_x^2)u_0$ does not change sign. The Camassa-Holm equation is remarkable since it combines complete integrability with the formation of singularties.

In the existence analysis of section 4, (4) will be considered, subject to the initial conditions

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{14}$$

Note that (4) contains additional information compared to (1), (2), since for bounded u, (3) is the unique bounded solution of the Poisson equation (2). The properties of the solution operator of the Poisson equation in a L^2-setting will be used, in particular the smoothing

$$\|\varphi[u]\|_{H^{k+2}(\mathbb{R})} \le \|u\|_{H^k(\mathbb{R})}, \qquad u \in H^k(\mathbb{R}), k \ge 0 \tag{15}$$

and the symmetry

$$\int_{\mathbb{R}} \varphi[u] v \, dx = \int_{\mathbb{R}} \varphi[v] u \, dx, \qquad u, v \in L^2(\mathbb{R}), \tag{16}$$

following from the evenness of the Green's function G.

The BP-equation (4) becomes the Benjamin-Ono equation when $\varphi[u]$ is replaced by $-2H[u_x]$, where H is the Hilbert transform:

$$H[u] = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{y-x} \, dy$$

The Benjamin-Ono equation arises in the study of long internal gravity waves in stratified fluids of great depth [2], [24]. It is a completely integrable Hamiltonian system [19] possessing multi-soliton solutions [3], [6]. There exists also the analogue of the inverse scattering method [1] and Bäcklund transformations [22]. Although the dispersive regularization by the Hilbert transform is weaker compared to KdV (cf. [20]), the dispersion is strong enough that the Benjamin-Ono equation has globally smooth solutions for initial data $u_0 \in H^k(\mathbb{R})$, $k \ge 3/2$ (see [17], [23]), and even for sublinearily growing initial data [15].

The BP-system (1),(2) is *Galilean invariant*, i.e. invariant under changes of the reference frame of the form

$$x \to x + x_0 + u_0 t, \qquad u \to u + u_0, \qquad \varphi \to \varphi - u_0.$$

Note that the potential transforms like a velocity. The Galilean invariance will simplify the travelling wave analysis in section 3.

The *dispersion relation* of the BP-equation (4) linearized at $u = c$ is given by

$$\frac{\omega}{k} = c + \frac{1}{1+k^2}, \tag{17}$$

with the frequency ω and the wave number k. The group velocities lie between c and $c+1$, the limits for large and small wave numbers, respectively. The existence of a finite limit for large wave numbers indicates that (4) does not have smoothing properties.

Finally, we look for *conservation laws*. Obviously, (4) can be written in conservation from:

$$u_t + \left(\frac{u^2}{2} - \varphi[u]\right)_x = 0 \tag{18}$$

As a consequence, $\int_{\mathbb{R}} u\,dx$ is conserved for weak solutions with sufficiently strong decay for $x \to \pm\infty$. Multiplication of (1) by $u = \varphi_{xx} - \varphi$ leads to the second conservation law

$$(u^2)_t + \left(\frac{2}{3}u^3 + \varphi^2 - \varphi_x^2\right)_x = 0. \tag{19}$$

Since we shall consider weak solutions based on the conservation law (18), the quantity $\int_{\mathbb{R}} u^2\,dx$ will only be conserved for smooth solutions. By the boundedness of the operator $\varphi_x[\cdot]$, we expect (as for the Burgers equation) nonuniqueness of weak solutions, which can be eliminated by an entropy condition. In section 4, weak solutions will be constructed satisfying the *entropy condition* (19) with the equality sign replaced by $\leq$. Thus, for weak solutions, the entropy $\int_{\mathbb{R}} u^2\,dx$ is nonincreasing. Note that, in contrast to the Burgers equation, not every convex function of u is an entropy density.

The *jump conditions* for entropic shocks with velocity s are those of the Burgers equation:

$$s = \frac{1}{2}\left(u_l + u_r\right), \qquad u_l > u_r, \tag{20}$$

where u_l and u_r denote the left-sided and, respectively, right-sided limit of u at the shock.

The BP-equation has an *Hamiltonian structure* similar to the Benjamin-Ono equation. The bi-Hamiltonian structure of the Camassa-Holm equation is completely destroyed by dropping the quadratic terms in (13). With the Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} \left(-\varphi[u]u + \frac{u^3}{3}\right) dx,$$

(4) can be written as

$$u_t + \left(\frac{\delta H}{\delta u}\right)_x = 0,$$

where $\frac{\delta H}{\delta u} = -\varphi[u] + \frac{u^2}{2}$ is the L^2-representation of the Frechet-derivative of H. Note that this relies on the symmetry property (16) of $\varphi[\cdot]$. Conservation of the quantity $\int_{\mathbb{R}} H(u)\, dx$ corresponds to the third local conservation law

$$\left(-\varphi u + \frac{u^3}{3}\right)_t + \left(\varphi_x \varphi_t - \varphi \varphi_{xt} - \left(-\varphi + \frac{u^2}{2}\right)^2\right)_x = 0$$

which (as (19)) can only be expected to hold for smooth solutions.

3 Travelling Wave Analysis

The results of this section should be compared to those of [10], [11] for different versions of the Euler-Poisson system. The qualitative similarities of the results have been one of the main motivations for this work.

By the Galilean invariance it suffices to consider only travelling waves with velocity 0, i.e. steady states. Travelling waves with velocity c are then found by adding the constant c to the velocity u (and $-c$ to the potential φ). After integration of the steady state version of (1) and using the result in (2), the steady state equations can be written as

$$uu_x = E, \tag{21}$$

$$E_x = \frac{u^2}{2} + u - d, \tag{22}$$

where we have used the notation $\varphi_x = E$ and d is the constant of integration. The system (21), (22) will be studied in the (u, E)-phase-plane. We shall also allow shocks (of course with velocity $s = 0$) satisfying the jump conditions (20):

$$-u_r = u_l > 0.$$

By (22), E is continuous across shocks.

Also worth mentioning is the line of singularities $u = 0$. In general, trajectories end (or begin) there with square root behaviour. By (21), smooth trajectories can only cross $u = 0$ through the origin of the (u, E)-plane. Our analysis will be restricted to $d > -1/2$, whence there are two stationary points

$$P_\pm = \begin{pmatrix} E_\pm \\ u_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \pm \sqrt{1+2d} \end{pmatrix}.$$

It is easily seen that u_- is always negative and a saddle. The second stationary point u_+ is negative and a center for $-1/2 < d < 0$. It becomes positive and a saddle for $d > 0$.

By separation of variable, a first integral of (21), (22) can be found:

$$A = \frac{E^2}{2} - \frac{u^4}{8} - \frac{u^3}{3} + \frac{du^2}{2} \tag{23}$$

Besides the stationary points, this familiy of curves (parametrized by A) has the origin as a critical point. Only away from the line $u = 0$, these curves can be seen as trjectories (with opposite orientation on opposite sides of $u = 0$).

Depending on the value of d, three generic cases of phase portraits occur.

Solitary Waves for $-1/2 < d \leq -4/9$:

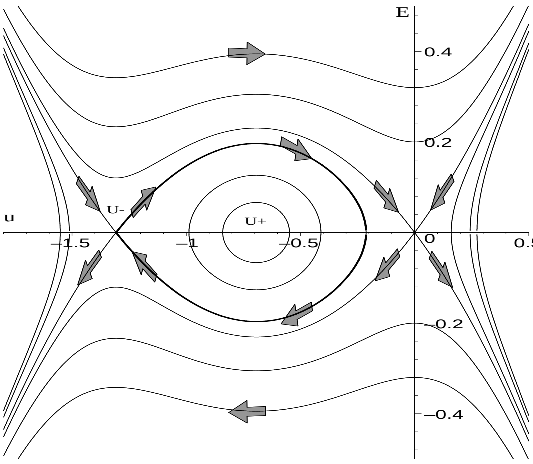

Fig. 1.1: Phase portrait for $-\frac{1}{2} < d \leq -\frac{4}{9}$

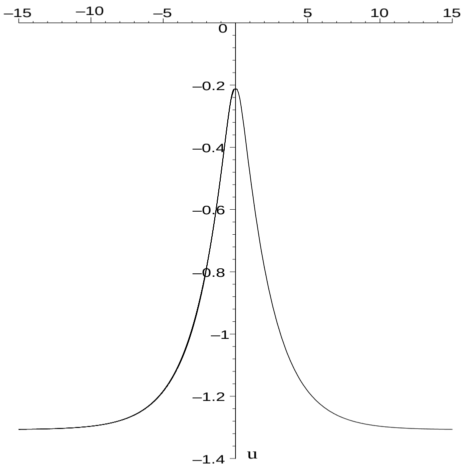

Fig. 1.2: Solitary wave

The phase portraits for $-1/2 < d < -4/9$ are characterized by a homoclinic orbit (pulse, solitary wave) connecting P_- to itself (see Fig. 1.1, Fig. 1.2). Its interior is filled with periodic solutions around P_+. These features are reminiscent of the KdV-equation. By the singularity, the origin is a point of nonuniqueness for the initial value problem. Taylor expansion shows a pair of smooth trajectories passing through the origin. An implicit formula for the solitary waves has already been calculated in [16] together with a numerical simulation of the soliton like interaction of two solitary waves.

In the critical case $d = -4/9$, the trajectories through the origin coincide with the stable and unstable manifolds of P_-. As a consequence of the nonuniqueness, we can switch from the unstable to the stable manifold at the origin, producing a nonsmooth solitary wave, reminicent of the peakon solutions of the Camassa-Holm equation. It can be computed explicitly:

$$u(x) = \frac{4}{3}\left(e^{-|x|/2} - 1\right)$$

Peaked periodic solutions for $-4/9 < d < 0$:

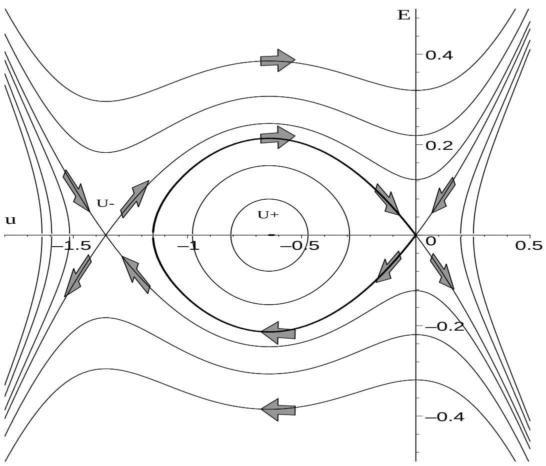

Fig. 2.1: Phase portrait for $-\frac{4}{9} < d < 0$

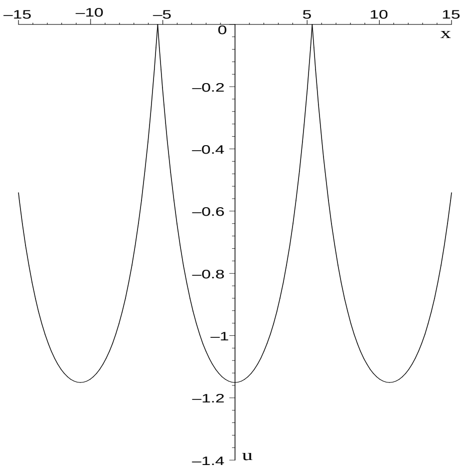

Fig. 2.2: Peaked periodic solution

In this case the solitary wave disappears and the trajectories passing through the origin connect to themselves (see Fig. 2.1). This closed curve in the left half plane corresponds to a peaked periodic solution (see Fig. 2.2). Again, these solutions can be computed explicitly. Taking the derivative of (21) and using (23) for the evaluation of u_x^2, we obtain (with A=0):

$$u_{xx} = \frac{u}{4} + \frac{1}{3}$$

The peaked periodic solution is given by

$$u(x) = \frac{4}{3}\left(\frac{\cosh(x/2)}{\cosh(p/2)} - 1\right)$$

for $-p < x < p$ and by periodic continuation with period $2p$. The length of the period is connected to the parameter d by

$$\cosh\left(\frac{p}{2}\right) = \left(1 + \frac{9}{4}d\right)^{-1/2}.$$

Heteroclinic connections for $d > 0$:

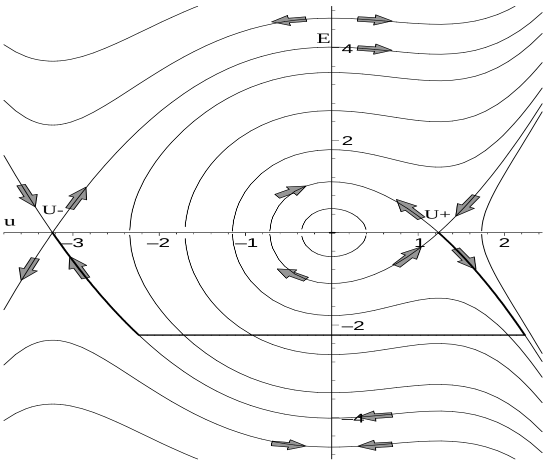

Fig. 3.1: Phase portrait for $d > 0$

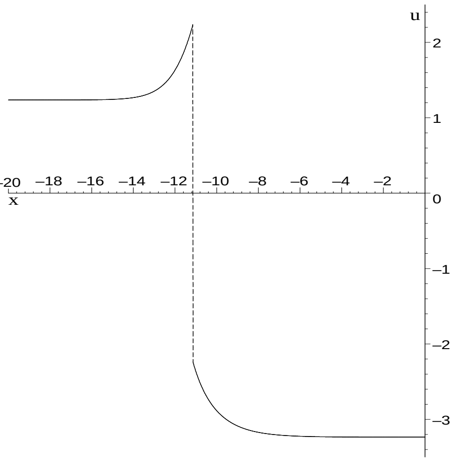

Fig. 3.2: Shock solution

In this case, the stationary points are saddles and lie on opposite sides of the line $u = 0$ (see Fig. 3.1). A heteroclinic connection (front wave) between them can be constructed using an entropic shock. There is a unique pair of points $P_l = (u_l, E_l) = (\sqrt{1+2d}, \sqrt{11/12+2d})$, $P_r = (-u_l, E_l)$, satisfying the jump conditions, with P_l lying on the unstable manifold of P_+ and P_r on the stable manifold of P_-. The u-component of the heteroclinic solution is depicted in Fig. 3.2.

Remark 3.1 *The question arises, if two arbitrary constant states $u_{-\infty}$, u_∞ can be connected by a front wave. The answer is negative. The set of admissible pairs $(u_{-\infty}, u_\infty)$ is constructed by shifting pairs $(u_+(d), u_-(d))$, $d > 0$, (exploiting the Galilean invariance). This leads to the requirement $u_\infty - u_{-\infty} > 2$.*

Transient Behaviour, Numerical Experiments

We have studied numerically the transient behaviour of solutions of the BP-equation (4). A MATLAB program was written employing a straightforward explicit discretization: In a first step, the Poisson equation is solved for given u at the old time step. A centered difference scheme is used on a bounded interval with boundary conditions $\varphi + u = 0$ at both ends. The result is used for the evaluation of the right hand side of (4). Alternatively, we used an implicit spectral method (cf. [7, Part II, Chapter 8]) and obtained very similar results. This spectral method is due to the use of FFT between 3-4 times faster but it applies only for spacially periodic situations.

The Burgers-flux term is discretized by the Lax-Friedrichs method. Time steps are chosen according to the CFL-condition. As initial data, linear ramps connecting two constant states are prescribed. Recalling remark 3.1, we are interested in the behaviour depending on the difference between the asymptotic states at $x = \pm\infty$.

Our results for downward ramps of height larger than 2 suggest the conjecture that the heteroclinic waves constructed above are attractive. For a typical example see Fig. 4.1 and Fig. 4.2. For a ramp with height 3 and the constant states lying symmetric with repect to $u = -1$, we observe numerical convergence to the stationary solution of type of Fig. 3.2. The development of shocks seems not to depend on the steepness of the ramp.

A completely different behaviour is observed for inital ramps with a height less than 2: in this case, there exists no stationary solution connecting the asymptotic states. The observed behaviour is reminicscent of the KdV-equation and shows typical dispersive effects with oscillations at the left side of a smoothed ramp. This is in accordance with the dispersion relation (17) showing that high frequency components travel with lower velocities. The example depicted in Fig. 4.3 and Fig. 4.4 involves an initial ramp of height 1. Again, long time convergence has been observed. However, this is a numerical artefact, since no steady state solution of the continuous problem with the qualitative behaviour shown in Fig. 4.4 exists.

4 Existence

In this section, the initial value problem

$$u_t + uu_x = \varphi_x[u], \qquad u(x,0) = u_0(x) \tag{24}$$

is considered, where the operator $\varphi[\cdot]$ is defined in (3).

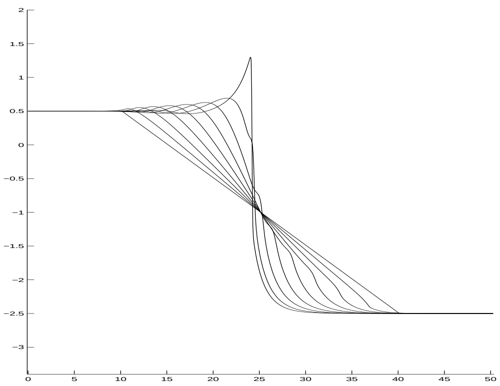

Fig. 4.1: Initial ramp: $0.5 \searrow -2.5$

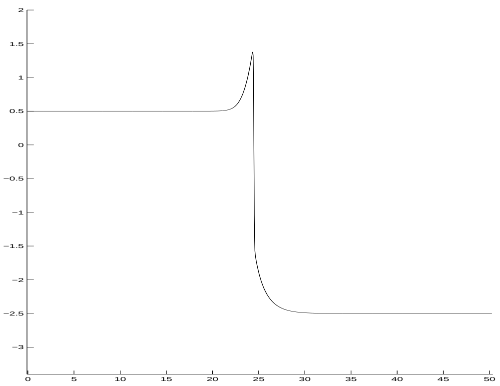

Fig. 4.2: Numerical stationary solution

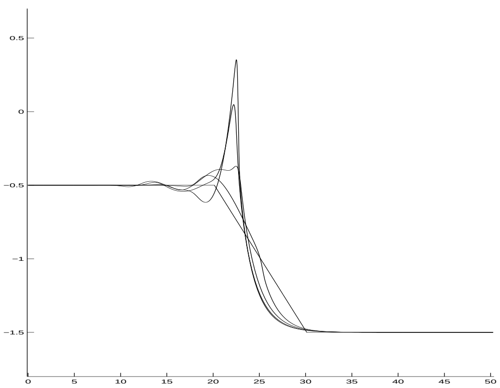

Fig. 4.3: Initial ramp: $-0.5 \searrow -1.5$

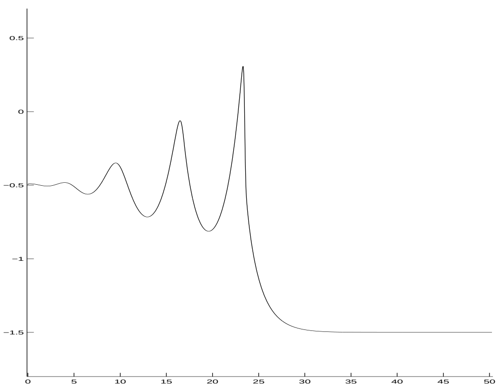

Fig. 4.4: Numerical quasistationary solution

Theorem 4.1 ***(Local strong solution)*** *Assume $u_0 \in H^k(\mathbb{R})$ with $k > \frac{3}{2}$. Then, there exists a time $T > 0$ and a unique solution*

$$u \in L^\infty\big((0,T); H^k(\mathbb{R})\big) \cap C\big([0,T]; H^{k-1}(\mathbb{R})\big)$$

of (24).

Proof: The proof is based on a contraction argument similar to [26, Section 16.1]. We define the iteration map S_T as follows: for any function $v \in B_T$ with

$$\begin{aligned} B_T \quad := \quad & \big\{ w \in L^\infty\big((0,T), H^k(\mathbb{R})\big) \cap C\big([0,T]; H^{k-1}(\mathbb{R})\big) : \\ & \sup\nolimits_{t\in[0,T]} \|w(\cdot,t)\|_{H^k(\mathbb{R})} \leq 2\|u_0\|_{H^k(\mathbb{R})} \big\} \end{aligned}$$

the image $S_T(v)$ is the unique solution u of

$$u_t + uu_x = \varphi_x[v], \qquad u(t=0) = u_0. \tag{25}$$

First, we show that S_T maps B_T into itself for T small enough. We apply ∂_x^α to (25) for $\alpha \le k$ and take the L^2-scalar product with $\partial_x^\alpha u$:

$$\tfrac{1}{2}\tfrac{d}{dt}\|\partial_x^\alpha u\|^2_{L^2(\mathbb{R})} \quad + \quad \underbrace{\int_{\mathbb{R}} \partial_x^\alpha(uu_x)\,\partial_x^\alpha u\,dx}_{A} \quad = \quad \underbrace{\int_{\mathbb{R}} \varphi_x[\partial_x^\alpha v]\,\partial_x^\alpha u\,dx}_{B} \tag{26}$$

The first factor in the integrand of A is differentiated by the product rule:

$$\partial_x^\alpha(uu_x) = u\,\partial_x^{\alpha+1}u + \sum_{l=1}^{\alpha}\binom{\alpha}{l}\partial_x^l u\,\partial_x^{\alpha+1-l}u$$

Accordingly, A is split into two parts which are estimated seperately:

$$\left|\int_{\mathbb{R}} u\,(\partial_x^{\alpha+1}u)(\partial_x^\alpha u)\,dx\right| = \frac{1}{2}\left|\int_{\mathbb{R}} u\,\partial_x(\partial_x^\alpha u)^2\,dx\right| \le \frac{1}{2}\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^2_{H^k(\mathbb{R})}. \tag{27}$$

For the second part of A, we use the Cauchy-Schwartz inequality and the interpolation estimate

$$\left\|\left(\partial_x^{l-1}f_x\right)\left(\partial_x^{\alpha-l}g_x\right)\right\|_{L^2(\mathbb{R})} \le c\left(\|f_x\|_{L^\infty(\mathbb{R})}\|g\|_{H^\alpha(\mathbb{R})} + \|f\|_{H^\alpha(\mathbb{R})}\|g_x\|_{L^\infty(\mathbb{R})}\right)$$

(see [26, Chapter 13, Proposition 3.6]) to obtain:

$$\left|\int_{\mathbb{R}} \partial_x^\alpha u \sum_{l=1}^{\alpha}\binom{\alpha}{l}\partial_x^{l-1}u_x\,\partial_x^{\alpha-l}u_x\,dx\right| \le c\|u\|^2_{H^k(\mathbb{R})}\|u_x\|_{L^\infty(\mathbb{R})}\,,$$

with some constant c only depending on k. By the Sobolev imbedding $W^{1,\infty}(\mathbb{R}) \hookrightarrow H^k(\mathbb{R})$ for $k > 3/2$, we calculate

$$|A| \le c\|u\|^3_{H^k(\mathbb{R})}. \tag{28}$$

For B, we apply the Cauchy-Schwartz inequality and (15):

$$|B| \le \|\varphi_x[\partial_x^\alpha v]\|_{L^2(\mathbb{R})}\|\partial_x^\alpha u\|_{L^2(\mathbb{R})} \le c\|v\|_{H^k(\mathbb{R})}\|u\|_{H^k(\mathbb{R})}. \tag{29}$$

Using (28) and (29) in (26) gives

$$\frac{d}{dt}\|u\|_{H^k(\mathbb{R})} \le c\left(\|u\|^2_{H^k(\mathbb{R})} + \|v\|_{H^k(\mathbb{R})}\right).$$

For T small enough, a comparsion principle shows $\|u(\cdot,t)\|_{H^k(\mathbb{R})} \leq 2\|u_0\|_{H^k(\mathbb{R})}$ for $0 \leq t \leq T$. Since $u \in C\big([0,T]; H^{k-1}(\mathbb{R})\big)$ is an obvious consequence of (25), we conclude that $S_T : B_T \to B_T$. In a second step, we prove S_T to be a strict contraction. Therefore, we consider two functions v_1 and v_2 in B_T and set $u_1 = S_T(v_1)$, $u_2 = S_T(v_2)$ and $u = u_1 - u_2$, $v = v_1 - v_2$. The difference of the equations for u_1, u_2 reads as

$$u_t + u(u_1)_x + u_2 u_x = \varphi_x[v] \qquad u(t=0) = 0.$$

We proceed similarily to (25) using B from (26):

$$\tfrac{1}{2}\tfrac{d}{dt}\|\partial_x^\alpha u\|^2_{L^2(\mathbb{R})} \ + \ \underbrace{\int_{\mathbb{R}} \partial_x^\alpha(u\partial_x u_1)\,\partial_x^\alpha u\,dx}_{A_1} \ + \ \underbrace{\int_{\mathbb{R}} \partial_x^\alpha(u_2 u_x)\,\partial_x^\alpha u\,dx}_{A_2} \ = B$$

In contrast to (27), the highest order term of A_1 is not bounded in terms of the $H^\alpha(\mathbb{R})$-norm. Therefore, we are obliged to reduce the order of differentation to $\alpha \leq k-1$ and estimate as above for the second part of A:

$$\begin{aligned} |A_1| &\leq c\|u\|_{H^{k-1}(\mathbb{R})}\big(\|\partial_x u_1\|_{L^\infty(\mathbb{R})}\|u\|_{H^{k-1}(\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R})}\|u_1\|_{H^k(\mathbb{R})}\big) \\ &\leq c\|u\|^2_{H^{k-1}(\mathbb{R})}\|u_1\|_{H^k(\mathbb{R})}, \end{aligned}$$

For A_2, we proceed as in (27)-(28).

$$\begin{aligned} |A_2| &= \left| \int_{\mathbb{R}} \left(u_2\,\partial_x(\partial_x^\alpha u)^2/2 + \partial_x^\alpha u \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_x^l u_2\,\partial_x^{\alpha-l} u_x \right) dx \right| \\ &\leq c\|u\|^2_{H^{k-1}(\mathbb{R})}\|u_2\|_{H^k(\mathbb{R})} \end{aligned}$$

Since $\|u_{1,2}\|_{H^k(\mathbb{R})} \leq 2\|u_0\|_{H^k(\mathbb{R})}$, this leads to

$$\frac{d}{dt}\|u\|_{H^{k-1}(\mathbb{R})} \leq c\left(\|u\|_{H^{k-1}(\mathbb{R})} + \|v\|_{H^{k-1}(\mathbb{R})}\right),$$

and the Gronwall lemma implies that for T small enough, S_T is a strict contraction with respect to the topology in $C\big([0,T]; H^{k-1}(\mathbb{R})\big)$. ■

Theorem 4.2 ***(Global weak solution)*** *Assume $u_0 \in BV(\mathbb{R})$. Then, there exists a gobal weak solution*

$$u \in L^\infty_{loc}([0,\infty); BV(\mathbb{R})) \tag{30}$$

of (24), satisfying the entropy condition

$$(u^2)_t + \left(\frac{2}{3}u^3 + \varphi^2 - \varphi_x^2\right)_x \leq 0 \tag{31}$$

in the distributional sense.

Proof: The proof is based on the viscosity method similary to [26]. Instead of (24), we consider the regularized equation

$$u_t + uu_x = \varphi_x[u] + \nu u_{xx} \tag{32}$$

with $\nu > 0$. Local existence of a unique smooth solution of the initial value problem for (32) with $u(t=0) = u_0 \in BV(\mathbb{R})$ can be shown by standard arguments. The next step is an L^1-stability result for (32). Let u_1, u_2 denote solutions of (32) with initial data $f_1, f_2 \in L^1(\mathbb{R})$. Then, the difference $v = u_1 - u_2$ satisfy

$$v_t + (wv)_x = \varphi_x[v] + \nu v_{xx}, \qquad v(t=0) = f_1 - f_2, \tag{33}$$

with $w = (u_1 + u_2)/2$. Let $\mathrm{abs}_\delta(\cdot)$ be a convex regularisation of the modulus, uniformly converging to $|\cdot|$ as $\delta \to 0$, and satisfying $|\mathrm{abs}'_\delta(v)| \leq 1$. Multiplication of (33) with $\mathrm{abs}'_\delta(v)$ and integration with respect to x leads to

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \mathrm{abs}_\delta(v)\,dx &= \int_{\mathbb{R}} wv\,\mathrm{abs}''_\delta(v) v_x\,dx - \int_{\mathbb{R}} \mathrm{abs}'_\delta(v)\varphi_x[v]\,dx \\ &\quad -\nu \int_{\mathbb{R}} \mathrm{abs}''_\delta(v)\,(v_x)^2\,dx. \end{aligned} \tag{34}$$

Since the function $\int_0^v s\,\mathrm{abs}''_\delta(s)\,ds$ converges to zero uniformly as $\delta \to 0$, we have for the first term on the right hand side of (34):

$$\int_{\mathbb{R}} wv\,\mathrm{abs}''_\delta(v) v_x\,dx = -\int_{\mathbb{R}} w_x \int_0^v s\,\mathrm{abs}''_\delta(s)\,ds\,dx \to 0 \qquad \text{as } \delta \to 0.$$

Boundedness of the operator $\varphi_x[\cdot] : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ can be shown easily. With the properties of abs_δ, we obtain from (34) in the limit $\delta \to 0$:

$$\frac{d}{dt}\|v\|_{L^1(\mathbb{R})} \leq c\|v\|_{L^1(\mathbb{R})}. \tag{35}$$

Analogously to [26, Chapter 16, Lemma 6.1], it can be shown that

$$\frac{d}{dt}\|v\|_{BV(\mathbb{R})} \leq c\|v\|_{BV(\mathbb{R})}.$$

holds for the solution of (32) as a consequence of (35). This is sufficient for proving that the solution of (32) is global and bounded in $L^\infty_{loc}([0,\infty); BV(\mathbb{R}))$ uniformly in ν. This again is sufficient for passing to the limit $\nu \to 0$ in (32). For the details we refer to [26]. The entropy inequality (31) follows from a standard argument. ■

5 Asymptotics and the Quasineutral Limit

In this section, we investigate the rescaled ($x \to x/\varepsilon, t \to t/\varepsilon$) BP-system

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon, \tag{36}$$
$$\varepsilon^2 \varphi_{xx}^\varepsilon = \varphi^\varepsilon + u^\varepsilon, \tag{37}$$

with $\varepsilon \ll 1$. In accordance with the terminology taken from the Euler-Poisson system, the limit $\varepsilon \to 0$ will be called the quasineutral limit. With the rescaled potential operator

$$\varphi^\varepsilon[u](x) = -\frac{1}{2\varepsilon}\int_{\mathbb{R}} \exp\left(-\frac{|x-y|}{2\varepsilon}\right) u(y)dy\,,$$

the initial value problem

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon[u^\varepsilon]\,, \qquad u^\varepsilon(t=0) = u_0\,, \tag{38}$$

will be compared to its formal limit

$$u_t^0 + (u^0+1)u_x^0 = 0\,, \qquad u^0(t=0) = u_0\,. \tag{39}$$

The limit is the inviscid Burgers equation for the unknown $u^0 + 1$. Even for smooth initial data its solution can develop shocks in finite time. The travelling wave analysis of section 3 can be seen as an attempt to approximate solution profiles in the neighbourhood of shocks of (39). The heteroclinic connections computed in section 3 are such profiles connecting two states u_l and u_r satisfying the jump conditions

$$s = \frac{1}{2}\left(u_l + u_r + 2\right), \qquad u_l > u_r\,.$$

However, these connections only exist for $u_l - u_r > 2$. For a better understanding of the situation, we expand the potential operator

$$\varphi^\varepsilon[u] = -u - \varepsilon^2 u_{xx} + O(\varepsilon^4)\,.$$

Thus, an $O(\varepsilon^4)$-approximation of (38) is given by the Korteweg-de-Vries equation (for the unknown $u+1$)

$$u_t + (u+1)u_x + \varepsilon^2 u_{xxx} = 0\,. \tag{40}$$

Actually, the Korteweg-de-Vries equation can be obtained as a formal limit of (38). If (38) is rescaled by

$$t \to \frac{t}{\varepsilon^2}\,, \qquad u^\varepsilon \to -1 + \varepsilon^2 U\,,$$

then the formal limit of the resulting equation for U is

$$U_t + UU_x + U_{xxx} = 0\,.$$

In analogy to the well known results concerning the limit as $\varepsilon \to 0$ of (40), we expect that in general the limits of solutions of (38) are weak limits, which do not satisfy the formal limiting equation (39). In our last result, however, we prove that — as long as the solution of the limiting equation remains smooth — it is the strong limit of the solution of (38). For $u_0 \in C^{0,1}(\mathbb{R})$ there exists a $T > 0$, such that the Burgers equation (39) has a solution $u^0 \in C\left([0,T];\; C^{0,1}(\mathbb{R})\right)$. Also, if $u_0 \in H^s(\mathbb{R})$, then $u^0 \in L^\infty((0,T);\; H^s(\mathbb{R}))$.

Theorem 5.1 *Assume $u_0 \in C^{0,1}(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > 1$. Then, for a time interval $(0,T)$ as mentioned above, the solutions of (38) and (39) satisfy*

$$\|u^\varepsilon - u^0\|_{L^\infty((0,T);L^2(\mathbb{R}))} = O\left(\varepsilon^r\right)\,, \qquad r = \min\{2, s-1\}\,.$$

Proof: Let $v = u^\varepsilon - u^0$ with inital data $v(t=0) = 0$. We subtract (39) from (38) to obtain an equation for v:

$$v_t + \left(\frac{v^2}{2} + u^0 v\right)_x = u^0_x + \varphi^\varepsilon[u^0_x] = \varepsilon^2 \varphi^\varepsilon_{xx}[u^0_x].$$

By taking the L^2-scalar product with v and by integration by parts, we calculate

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} \frac{v^2}{2}\, u^0_x\, dx = \varepsilon^2 \int_{\mathbb{R}} v\, \varphi^\varepsilon_{xx}[u^0_x]\, dx,$$

which implies

$$\frac{d}{dt}\|v\|_{L^2(\mathbb{R})} \le \frac{1}{2}\|u^0_x\|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} + \|\varepsilon^2 \varphi^\varepsilon_{xx}[u^0_x]\|_{L^2(\mathbb{R})}. \tag{41}$$

With the Fourier transform $\widehat{u^0}(k,t)$ of $u^0(x,t)$, the last term can be estimated by

$$\|\varepsilon^2 \varphi^\varepsilon_{xx}[u^0_x]\|_{L^2(\mathbb{R})} = \left\| \frac{\varepsilon^2 |k|^3 |\widehat{u^0}|}{1+\varepsilon^2 k^2} \right\|_{L^2(\mathbb{R})} \le \sup_{k\in\mathbb{R}} \frac{\varepsilon^2|k|^3}{(1+\varepsilon^2k^2)(1+k^2)^{s/2}} \|u^0\|_{H^s(\mathbb{R})}\,.$$

The factor on the right hand side is obviously $O(\varepsilon^2)$ for $s \ge 3$. For $s < 3$, it can be estimated by

$$\frac{\varepsilon^2|k|^3}{(1+\varepsilon^2k^2)|k|^s} = \varepsilon^{s-1}\frac{|\varepsilon k|^{3-s}}{1+|\varepsilon k|^2} = O(\varepsilon^{s-1})\,,$$

and, thus,

$$\|\varepsilon^2 \varphi^\varepsilon_{xx}[u^0_x]\|_{L^2(\mathbb{R})} \le c\varepsilon^r \|u^0\|_{H^s(\mathbb{R})}\,.$$

The statement of the theorem is now a direct consequence of the Gronwall lemma applied to (41). ■

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