On the Clock of the Combinatorial Clock Auction

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Abstract

The Combinatorial Clock Auction (CCA) has been frequently used in recent spectrum auctions. It combines a dynamic clock phase and a one-off supplementary round. The winning allocation and the corresponding prices are determined by the VCG rules. These rules should encourage truthful bidding, whereas the clock phase is intended to reveal information. We inquire into the role of the clock when bidders have lexicographic preferences for raising rivals’ costs. We show that the CCA does not have efficient equilibria where the clock phase fully reveals information. In addition, if there is substantial room for information revelation, that is, if the uncertainty about the final allocation is large, all equilibria are inefficient. Qualitative features of our equilibria are in line with evidence concerning bidding behavior in some recent CCAs.

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1 Introduction

In recent years, many regulators around the world have chosen the Combinatorial Clock Auction (CCA) to allocate telecommunication spectrum. The CCA has partially replaced the older Simultaneous Ascending Auction (SAA) for two reasons. First, in the SAA bidders may have an incentive to strategically reduce demand in order to establish a “reasonable” allocation at low prices. The sophisticated design of the CCA should overcome this issue as it incorporates (i) a generalized second-price (Vickrey) rule providing bidders an incentive to bid truthfully (Cramton, 2013), while (ii) a clock phase should facilitate “price and package discovery” (Ausubel, Cramton and Milgrom, 2006). Second, unlike the SAA, bidders in the CCA are able to express bids for packages. This is deemed to be important as modern spectrum auctions allocate multiple units where bidders may value the units as complements. If that is the case the SAA, but not the CCA, suffers from the well-known exposure problem. The focus of this paper is the first issue: is it the case that the CCA provides bidders with an incentive to bid truthfully and that the clock phase facilitates “price and package discovery”? 

The CCA is a dynamic version of the well-known Vickrey-Clarke-Groves (VCG) mechanism and consists of two integrated phases.1 In the first clock phase bidders express their demand on packages at given prices in every round. If for a certain good demand is larger than supply in a given round, then the price for that good increases in the next round. The clock phase ends when demand is not larger than supply for all the auctioned goods. Importantly, no goods are allocated and no prices are determined at the end of the clock phase. Instead, the clock phase imposes constraints on the admissible bids bidders are allowed to place in the second supplementary phase. In that phase, bidders can bid on as many additional packages as they like and they may raise bids on packages they have bid on in the clock phase. At the end of the supplementary phase, goods are allocated and prices are determined. The regulator uses all the bids from the clock phase

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1In practice there is a third phase - the assignment phase. In this phase generic packages are allocated. We abstract away from this phase since it does not effect our analysis.
and the supplementary phase to determine the value maximizing combination of bids. The auctioneer uses the Vickrey-pricing rule to determine the prices winners have to pay (e.g. Milgrom, 2004).

Without the clock phase the CCA reduces to the VCG mechanism. As the number of packages is an exponential function of the number of commodities, bidders in a VCG auction may need to consider bidding on a very large number of packages. In particular, if the uncertainty concerning competitors is large, they may have a fairly limited idea about the package they may eventually win and at which price. The clock phase is meant to reveal this kind of information. Bidders can then focus their bidding in the supplementary round on the packages that may still be winning.

Under standard preferences, truthful bidding in the clock and supplementary phase is indeed an equilibrium. If bidders bid truthfully the outcome is efficient. However, truthful bidding is not a strict equilibrium as bidders may be indifferent across all their permissible bids in the supplementary round (Levin and Skrzypacz, 2016). To eliminate the payment relevant indifferences we consider bidders that ceteris paribus prefer outcomes where competitors pay more. We model this objective as a secondary dimension in a lexicographic way.

Our first main result is that an efficient equilibrium where the clock phase fully reveals bidders’ types does not exist if bidders have a lexicographic preference to raise rivals’ cost. This result implies that the CCA imposes a fundamental trade-off between efficiency and information revelation in the clock phase. The trade-off follows from the fact that if bidders bid truthfully in the clock phase, the clock reveals information about other bidders’ types. Bidders would like to use this information to maximally raise rivals’ cost by placing bids in the supplementary phase on large packages that they know cannot be winning. The stronger their competitors, the more they can raise their prices. The rules of the CCA are such that bidders are only able to raise rivals’ cost if they expand demand in the clock phase as this relaxes the constraints on the admissible supplementary phase bids. Predicting that the clock phase eventually will fully reveal information, bidders can expand demand in the early phase of the clock without the risk of affecting the final allocation. Knowing that competitors are able and inclined to raise their cost if their types are fully revealed,
stronger bidders have an incentive to pool with weaker types in the clock phase.

This result is best understood from the perspective of the ratchet effect from the dynamic principal-agent literature (Laffont and Tirole, 1988). In that literature, an agent may have an incentive not to reveal her type to a principal if the principal could use that information to extract more surplus from the agent in future interactions. In our case, knowing their competitors are strong, a bidder (by bidding more aggressively in the supplementary phase) may increase the price competitors have to pay beyond what they would do if the competitor type is unknown. Predicting this possibility for exploitation, stronger bidders prefer to pool with weaker types.

The intuition for our first main result differs in two dimensions from the traditional ratchet effect. First, unlike the principal-agent model, the roles of bidders in an auction are symmetric to one another so that each bidder is both the object of and the initiator of surplus extraction. Second, the extent to which bidders are able to raise rivals’ cost in the supplementary round is not exogenously given, but endogenously determined by their behavior in the clock phase. Thus, bidders will only be able to raise rivals’ cost if they expand demand in the clock phase.

The result that fully revealing efficient equilibria do not exist does not rule out the existence of efficient equilibria. Even with lexicographic preferences to raise rivals’ cost efficient equilibria can exist. We present examples of efficient equilibria. In any of these equilibria, to be able to raise rivals’ cost, bidders need to relax the constraints their clock phase bidding imposes on the bids they are permitted to place in the supplementary phase. They will do so by demanding the full supply (even if prices are such that truthful bidding would tell them to reduce demand). We construct a clock-pooling equilibrium where at a certain clock price all bidders drop demand from the full supply to their truthful demand. If the ex ante uncertainty concerning rivals’ types is relatively small, then there exists a range of prices such that weak bidders still have a positive demand at these prices, while the truthful demand of strong bidders is smaller than their proportional share of the available supply. Thus, when all bidders drop demand the clock stops.

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2This is in line with, for example, the Austrian 2013 auction where (as we mention below) bidders were bidding very aggressively in the clock phase.
immediately. In such an equilibrium, there is no price or package discovery whatsoever. This clock phase development allows all bidders in the supplementary round to bid their true marginal values over the relevant shares. Accordingly, the final allocation is efficient. We also show that any efficient equilibrium of the CCA, and not only the clock-pooling equilibria that we construct, involves demand expansion in the clock phase.

Our second main result is that if, with two bidders, the uncertainty concerning the competitor’s type is sufficiently large, all equilibria of the CCA are inefficient. Efficiency requires that at relatively high clock prices, weak bidders drop their demand in the clock phase such that if all bidders are relatively weak the clock phase ends. However, if some bidders turn out to be relatively strong, the clock phase continues and these bidders demand more than half of the full supply. A relatively weak bidder is then able to infer that their competitors are relatively strong. This learning effect creates the opportunity for relatively weak bidders to make their supplementary round behavior conditional on the price at which the clock phase stops. Knowing (some) competitors are strong, they can raise rivals’ cost more (without running the risk of winning more than they would like to win). When the ex ante uncertainty concerning competitors’ types is large, strong bidders find it optimal to reduce their demand towards the end of the clock phase in a way that prevents them to express their true valuation for all possible allocations in the supplementary round. This prevents weak bidders to raise their cost, but it creates an inefficient final allocation. As we will also show that independent of the degree of uncertainty regarding competitors’ types, the static VCG mechanism always has efficient equilibria, we claim that it is the clock phase that creates this inefficiency.

We also present two other interesting results. First, in any efficient equilibrium, the clock ends with excess supply with positive probability. This is in stark contrast with the fact that the clock would always end with market clearing under truthful bidding in our framework. We also show that, despite what many observers of the CCA have argued, this

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3See, e.g., Levin and Skrzypacz (2016, remark 2 on page 2542) where they observe that “If we allowed bidder 2 to create excess supply at the end of the clock phase, she could increase bidder 1 payment even more. ... Such extreme predatory behavior is even more difficult to execute and even more risky for player 2 than what we describe. Moreover, analyzing equilibria in this case is difficult, so we maintain the assumption
does not limit the possibility of bidders to raise rivals’ cost. Second, there are equilibria where two almost identical bidders pay significantly different amounts for an almost identical share (half) of the spectrum.

The lexicographic modeling of bidders’ preference to raise rivals’ costs implies that if two bidding strategies yield the same expected surplus to a bidder, the bidder chooses the strategy where rivals pay more.\textsuperscript{4} The raising rivals’ cost motive may stem from (i) principal-agent issues within a firm (bidder)\textsuperscript{5} or from (ii) the fact that (in spectrum auctions) bidders face weaker competitors in the market after an auction if competitors have paid more for their licenses. If firm A makes B pay more for spectrum, B’s credit rating will fall and its cost of capital will go up, weakening its strategic position. Milgrom (2004) and Cramton and Ockenfels (2014) mention fairness as a reason for why bidders may want to raise rivals’ cost. Of course, if bidders repeatedly interact in the same market, bidders may be able to collude as all may realize they are better off if they do not raise each others’ cost. In that case, it is more likely bidders behave to reduce (rather than to raise) rivals’ cost.

The raising rivals’ cost motive has become a concern in designing auctions.\textsuperscript{6} After the 2013 auction the Austrian regulator RTR attributed the

\begin{footnotesize}
\textsuperscript{4}Our analysis with lexicographic preferences can be considered a robustness check on the equilibria under standard preferences: equilibria under our preferences are also equilibria under standard preferences, but the reverse does not necessarily hold true.

\textsuperscript{5}In spectrum auctions, given the large amount of uncertainty concerning future technological developments and uptake of data services, it is difficult for bidders to evaluate what the spectrum is really worth. Valuations are highly subjective. Accordingly, if a bidder wants to have a more objective evaluation measure of its bidding team’s performance, it might be better to evaluate performance relative to other bidders, than it is to evaluate relative to the uncertain and subjective own valuation.

\end{footnotesize}
high revenue to overly aggressive behavior by bidders: during the clock phase, bidders were bidding very aggressively and the majority of the supplementary bids were on very large packages that had a low probability of winning but played a crucial role in determining other bidders’ prices. The fact that payments in the Austrian auction were essentially the same as the final clock prices, is a clear signal of aggressive bidding and is an empirical indication of the clock-pooling equilibrium: with Vickrey pricing and “downward sloping demand”, one would not expect marginal and average prices to be identical. Moreover, in a consultation document on the award of the 2.3 GHz and 3.4 GHz bands, the British regulator Ofcom (2014, p. 38, 6.73-6.77) explicitly mentions the possibility of price driving by placing “risk-free bids” in the supplementary phase as a problematic aspect of the CCA. Some of the potential bidders’ responses share this concern (e.g. BT, 2015). Although, none of these arguments for raising rivals’ cost implies that bidders should have a lexicographic preference for doing so, lexicographic preferences are a useful modeling approach to inquire into the effects of this motive.

We consider an auction where $n$ bidders compete to get a share of a perfectly divisible good. Bidders need to get a (small) minimum share to get positive utility. Once utility is positive, marginal utility is decreasing. This model is a generalization of Levin and Skrzypacz (2016) who consider an auction with two bidders whose utility function is quadratic and positive for every positive share. We use their specific set-up to illustrate some of equilibrium features in specific examples. The main difference with Levin and Skrzypacz (2016) is not that we use a more general set-up, but rather that they restrict themselves to strategy sets that are limited to linear proxy strategies for at least one of the bidders. When considering the implications of bidders wanting to raise rivals’ cost, they show that the optimal reaction is to reduce demand in the clock phase assuming that this will not affect the ability to raise rivals’ cost. We show that when bidders’ strategy sets are unrestricted, there exist efficient equilibria where bidders do not reduce demand even if competitors raise rivals’ cost. Moreover, in inefficient equilibria where bidders do reduce demand, there is still an initial phase of demand expansion so that linear proxy strategies are not optimal.
This paper contributes to the growing literature that explores real-world auction mechanisms. Ausubel et al. (2014) analyze the discriminatory and the uniform price auction in a similar framework. Goeree and Lien (2014) derive equilibria for the SAA and find that the exposure problem is indeed problematic for efficiency and revenue. However, as the number of items grows large, outcomes converge to VCG outcomes. Bichler et al. (2013) report experimental evidence on the CCA and present a simple example in which one bidder submits a spiteful bid. Gretschko et al. (2017) discuss why bidding can be complicated in a CCA. Ausubel and Baranov (2014) discuss the evolution of the CCA. A variant of the CCA has first been suggested by Ausubel et al. (2006) and further developed in Cramton (2013).

The rest of this paper is organized as follows. Section 2 describes the different auction models and the environment we analyze. To provide a benchmark Section 3 analyzes the VCG mechanism. We show that under standard preferences, iterated elimination of weakly dominated strategies always results in an efficient outcome, but it leaves the bids of weak types on large shares undetermined. The lexicographic preference we use imposes that bids on large shares are chosen to raise rivals’ cost. Section 4 proves our first main result that there do not exist efficient equilibria of the CCA where the clock phase fully reveals rivals’ types. Section 5 presents two examples of efficient equilibria where the clock either reveals no information to bidders, or only partial information. This section also shows that efficient equilibria are characterized by demand expansion in the first phase of the auction. Section 6 presents our second main result, namely that if there are two bidders and the uncertainty concerning the competitor’s type is large, the CCA does not have efficient equilibria. Section 7 provides an example of an inefficient equilibrium involving demand reduction, while Section 8 concludes with a discussion where we also discuss the relevance of our paper for interpreting real-world auctions. All proofs are in the appendix.

2 Model and Auction Rules

We consider auctions where \( n \) bidders compete to get a share of a perfectly divisible good that is in unit supply. Bidder \( i \)’s privately known type is denoted by \( a_i \). A bidder’s type is randomly drawn from an atom-less and
commonly known distribution with support \([a, \bar{a}]\). The set of type profiles \(a = (a_1, \ldots, a_n)\) is denoted by \(A = [a, \bar{a}]^n\). The utility a bidder of type \(a_i\) derives from acquiring a share \(x\) is denoted by \(U(a_i, x)\). To allow for a form of complementarity (such that a potential exposure problem exists) one may assume there is an \(z \geq 0\) such that \(U(a_i, x) = 0\) for all \(x \leq z\). Not to complicate the notation, in what follows we will assume \(z = 0\), but all the results of the paper continue to hold if \(z\) is positive, but small enough. The utility function \(U(a_i, x)\) is strictly increasing in \(a_i\) and \(x\), twice continuously differentiable, and concave. The marginal utility is increasing in \(a_i\), i.e., \(\partial^2 U(a_i, x)/\partial a_i \partial x > 0\) for \(x > 0\) and non-negative for \(x = 0\). When convenient we write \(U_i(x)\) instead of \(U(a_i, x)\). Throughout the paper we denote utility and bidding functions with capital letters and the respective derivatives with small letters. For example, we write \(U_i\) for the utility function and \(u_i\) for marginal utility. Also, \(U = U(a, \cdot)\) denotes the utility function of the weakest possible bidder.

Besides these standard preferences, the bidders have a spite motive. We model this spite motive in a lexicographic way. In the first dimension, bidders maximize their surplus from the auction and in the second dimension they maximize the sum of the payments of all other bidders. This spite motive is relatively weak since bidders do not want to harm other bidders if this implies getting a lower surplus themselves. Introducing a spite motive in a lexicographic manner resolves indifferences concerning auction outcomes in favor of outcomes that harm other bidders most.\(^7\) With only two bidders (one competitor), the spite motive is easy to model, but with more than two bidders there are some issues that are better dealt with after the rules of the VCG mechanism are presented. That is why the formal description of the spite motive is postponed until the last part of this section.

For every type profile \(a \in A\), we define the efficient allocation \(x^*\) as

\[
x^*(a) \in \arg \max_x \sum_{i=1}^n U(a_i, x_i) \text{ s.t. } \sum_{i=1}^n x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i.
\]

From the above assumption, a few results are immediate. Since the utility

\(^7\)When we talk about efficiency we talk about efficiency in the first dimension of the preferences only.
functions are strictly increasing and concave, there exists a unique efficient allocation, which may involve some bidders not getting anything. As the objective function of the constrained maximization problem is supermodular in \((a_i, x_i)\), Topkis’ Monotonicity Theorem implies that bidder \(i\)’s efficient share \(x_i^*(a) = x_i^*(a_i, a_{-i})\) is non-decreasing in \(a_i\), and hence, it is non-increasing in \(a_j\). It follows that for each type \(a_i\) there exists a lowest possible efficient share \(\min_{a_{-i}} x_i^*(a_i, a_{-i}) = x_i^*(a_i, \bar{a}, \ldots, \bar{a}) = \bar{x}_i\), and a largest possible efficient \(\max_{a_{-i}} x_i^*(a_i, a_{-i}) = x_i^*(a_i, \underline{a}, \ldots, \underline{a}) = \underline{x}_i\). Concavity of \(U\) implies that the allocation \((1/n, \ldots, 1/n)\) is efficient for any symmetric type profile. As a consequence, for types \(a < a_i < \bar{a}\) we have that \(\bar{x}_i < 1/n < \bar{x}_i\). In any efficient allocation, the lowest type will never win more than \(1/n\), while a strong type \(\bar{a}\) will not win less than \(1/n\). Berge’s Maximum Theorem implies that \(x^*(a)\) is continuous in \(a\). Hence, for any \(x \in [\bar{x}_i, \bar{x}_i]\) there exists a type profile \(a_{-i}\) such that \(x = x^*(a)\). Finally, we have that \(u(a_i, \bar{x}(a_i))\) is non-decreasing in \(a_i\).\(^8\) When there is no danger of causing confusion, we sometimes drop the subscript \(i\) in \(x_i^*\).

The value function of the maximization problem defining the efficient allocation is \(V(a) = \sum_{i=1}^n U_i(x_i^*)\). It is non-decreasing in \(a_i\) for all \(i\), since, by the envelope theorem, \(\partial V(a)/\partial a_i = \partial U(a_i, x_i)/\partial a_i \geq 0\). Let \(V_i(\underline{a})\) denote the minimal value of the efficient allocation for type \(a_i\), i.e. \(V_i(a) = V(a_i, \underline{a}, \ldots, \underline{a})\).

The set-up of Levin and Skrzypacz (2016) is a special case of the above description. Levin and Skrzypacz (2016) consider environments where the divisible good is allocated between two bidders \((n = 2)\) that have a strictly increasing quadratic utility function of the form

\[
U(a_i, x) = a_i x - \frac{b}{2} x^2,
\]

with \(a \geq b > 0\) and \(x \in [0, 1]\). The condition \(a \geq b\) makes the utility function increasing in \(x\) for all types. Levin and Skrzypacz (2016) adopt the assumption \(\bar{a} - \underline{a} < b\), which guarantees that the efficient allocation

\(^8\)This can be seen as follows. Let \(a_i' > a_i\), so \(\bar{x}_i > \bar{x}_i\). Suppose \(\bar{x}_i = \bar{x}_i\). Then clearly \(u(a_i', 1) > u(a_i, 1)\). If \(\bar{x}_i = 1\), but \(\bar{x}_i < 1\), then \(u(a_i', 1) > u(\underline{a}, 0) \geq u(\underline{a}, 1 - x_i)/(n - 1) = u(a_i, \bar{x}_i)\), by decreasing marginal values and necessary conditions of efficiency. If \(1 > \bar{x}_i\), then efficiency requires \(u(a_i, \bar{x}_i) = u(\underline{a}, (1 - \bar{x}_i)/(n - 1))\) and \(u(a_i', \bar{x}_i) = u(\underline{a}, (1 - \bar{x}_i)/(n - 1))\). Since \((1 - \bar{x}_i)/(n - 1) \geq (1 - \bar{x}_i)/(n - 1)\), marginal values imply \(u(\underline{a}, (1 - \bar{x}_i)/(n - 1)) \leq u(\underline{a}, (1 - \bar{x}_i)/(n - 1))\).
is always in the interior of $[0, 1]$ as $u_i(0) > u_j(1)$, $j \neq i$. Note that in our general set-up we do not make such an assumption and that the efficient allocation may be a corner solution. The efficient share of bidder $i$ is then

$$x^*_i(a_i, a_j) = \frac{a_i - a_j + b}{2b}.$$  

We will use this special case in our examples in Sections 5 and 7.

**VCG Rules**

In the VCG mechanism, every bidder submits a bidding function $S_i : [0, 1] \rightarrow \mathbb{R}_+$. As bidding functions depend on the type of a player, we also use $S(a_i, x)$ to denote a bidder’s bid on quantity $x$ when he is of type $a_i$. The auctioneer chooses the allocation $x$ with

$$x \in \arg \max_x \sum_{i=1}^n S_i(x_i) \text{ s.t. } \sum_{i=1}^n x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i.$$  

If two allocations solve the maximization problem, the auctioneer implements the allocation in which the distance to the allocation where all bidders get equal shares is minimized. Bidder $i$ gets $x_i$ and pays the VCG price $\max \sum_{j \neq i} S_j(y_j) - S_j(x_j)$ subject to $\sum_{j \neq i} y_j \leq 1$, the opportunity cost he imposes on the other bidder. If, in the specification of Levin and Skrzypacz (2016) with $n = 2$, the final allocation is $(x, 1 - x)$, then bidder $i$’s surplus from the auction is given by

$$U_i(x) - \max_y S_j(y) + S_j(1 - x).$$  

**CCA Rules**

The CCA is a two stage auction. In a first, clock phase bidders express a (weakly) decreasing demand at a certain price. In the second, supplementary phase a VCG auction is held where bidders submit many bids subject to so-called activity rules that are described below. Put differently, the clock phase elicits a demand function, whereas in the supplementary phase bidders submit an inverse demand function. Activity rules insure the consistency of the two functions.
We analyze a CCA where bidders do not receive any information concerning total demand in the clock phase.\textsuperscript{9} In this case, bidders can only condition their demand in the clock phase on the price, and cannot condition their demand on what rivals have demanded in previous clock rounds, making this type of auction easier to analyze.\textsuperscript{10} Thus, bidder $i$’s action in the clock phase is a weakly decreasing demand function $x_i : \mathbb{R}_+ \rightarrow [0, 1]$ that maps prices to demand. Bidders are not allowed to increase their demand during the auction. The clock phase begins at an initial price $p_0 = 0$ and the clock price is increased continuously as long as there is excess demand. The clock phase stops at $p$ if excess demand is smaller than or equal to zero, i.e., if $\sum_{i=1}^{n} x_i(p) \leq 1$.

In the supplementary phase, bidders submit bidding functions $S_i : [0, 1] \rightarrow \mathbb{R}_+$. The choice of the function $S_i$ is constrained by three activity rules. First, clock bids remain valid, that is, if bidder $i$ demanded $x$ at clock price $p$, then it has to be the case that $S_i(x) \geq p \cdot x$. This is a minimal requirement to make clock bidding meaningful. Second, supplementary bids must satisfy the so-called final cap, i.e., $S_i(x) \leq S_i(\tilde{x}_i) + \tilde{p}(x - \tilde{x}_i), x \neq \tilde{x}_i$, where $\tilde{p}$ is the final clock round price and $\tilde{x}_i$ is bidder $i$’s demand in the final clock round. The final cap rule essentially requires that supplementary round bids satisfy the axiom of revealed preference with respect to the final clock round behavior. These constraints bound the supplementary bidding function from above. If the clock ends with market clearing, then the final cap implies that the final clock allocation is the final allocation. Finally, if in the clock phase bidder $i$ was demanding $x$ at a price $p$, then for any $x' > x$, bidder $i$ cannot express an incremental bid for $x'$ in the supplementary round that is larger than $p$, i.e., $S_i(x') \leq S_i(x) + p(x' - x)$. For

\textsuperscript{9}In practice, CCAs have different regimes concerning the information that is released to the bidders in the clock phase. In one regime, bidders are only informed about the fact that there is still excess demand and that the clock phase continues. In another regime, bidders are informed about total demand at every clock price. The first regime was used in the first part of the Austrian auction and seems to be favored in case there is some suspicion that collusion between bidders may be something to worry about.

\textsuperscript{10}In the consultation document on the award of the 2.3 GHz and 3.4 GHz bands, Ofcom (2014) proposes to use either the CCA or the SAA without demand disclosure. In a reaction for Hutchinson 3G, Power Auctions LLC (2015) claims that a dynamic auction with no demand disclosure is basically a sealed-bid auction. We show, however, that the equilibria that can be sustained in a CCA without demand disclosure during the clock phase differ from the equilibria of the VCG.
differentiable bidding functions \( s_i(x') \leq p \) must hold. Levin and Skrzypacz (2016) call this constraint the ‘local revealed preference rule,’ which is also known as the relative cap. The relative cap imposes constraints on the slope of the bidding function. A bid on \( x' \) cannot be larger than the area under the expressed clock demand curve.

Given the bids in the clock and the supplementary phase, the auctioneer uses the same rules as described above for the VCG mechanism to compute the final allocation and individual CCA prices.\(^{11}\)

Bidders use the information about the clock development to update their beliefs about the possible types of the rival bidders. Even though no direct information is revealed to bidders, bidders may infer information about their competitors’ type as, given the equilibrium strategies, the length of the clock phase may reveal competitors’ types. We denote by \( \mathcal{A} \) the support of the posterior of the other bidders’ type distribution. If a bidder does not learn anything about the other bidders types, then \( \mathcal{A} = [a, \overline{a}]^{n-1} \). On the contrary, if there are only two bidders and the equilibrium is such that the final clock price allows bidder \( i \) to learn the rival’s type, then the posterior is a singleton: \( \mathcal{A} = \{a_j\} \). The set \( \mathcal{A}(p) \) denotes the support of the posterior if the clock ends at price \( p \).

Preferences for raising rivals’ costs and equilibrium notion

We can now formalize the spite motive. We start with the description of the spite motive in the context of the VCG auction and then extend it to the CCA. A bidder who faces a single rival bidder simply wants to raise this bidder’s VCG price. With more than two bidders there are many ways to maximize the sum of other bidders’ VCG prices. We choose to use a weak and cautious formulation of the spite motive. Given strategy profile \( S_{-i} \), bidder \( i \) prefers strategy \( \hat{S}_i \) over strategy \( S_i \) if and only if \( \hat{S}_i \) yields a strictly higher primary expected utility (denoted by \( U(a_i, x) \)) than \( S_i \), or the primary expected utility is the same, but \( \hat{S}_i \) leads to a higher sum of VCG prices for the type profile that minimizes the value of the

\(^{11}\)We do not consider the “core-selecting” elements in the pricing rule of real-world CCAs auctions (see, e.g. Day and Milgrom (2008), Day and Cramton (2012), and Erdil and Klemperer (2010), as well as Goeree and Lien (2016) and Ausubel and Baranov (2013)).
implemented allocation. Formally, let $\hat{x}(a)$ be the allocation implemented by $(\hat{S}_i, S_{-i})$ and let $x(a)$ be the allocation implemented by strategy profile $S$. Let $\hat{a}_{-i} \in \arg\inf_{a_{-i} \in A} \hat{S}_i(\hat{x}_i(a_i, \hat{a}_{-i})) + \sum_{j \neq i} S(\hat{a}_j, x_j(a_i, \hat{a}_{-i}))$ and $a_{-i} \in \arg\inf_{a_{-i} \in A} \sum_j S(\hat{a}_j, x_j(a_i, a_{-i}))$ be the two type profiles that minimize the value of the implemented allocation. The strategy $\hat{S}_i$ is preferred to $S$ in the spite dimension if

$$\sum_{j \neq i} \sup_{\sum_{k \neq j} y_k \leq 1} \hat{S}(a_i, y_i) + \sum_{k \neq j, i} S(\hat{a}_k, y_k) - S(a_i, \hat{x}_i) - \sum_{k \neq j, i} S(\hat{a}_k, \hat{x}_k) \geq \sum_{j \neq i} \sup_{\sum_{k \neq j} y_k \leq 1} \sum_{k \neq j} S(a_k, y_k) - \sum_{k \neq j} S(a_k, x_k).$$

In this definition the sum of VCG prices is raised for one particular type profile. This raises the VCG price also for other type profiles, but maybe not as much as is deemed possible given the belief $A$ about other bidders’ types.\(^{12}\)

In the CCA a strategy consists of a clock demand function $x_i$ and a supplementary bidding function $S^p_i$ for every possible final clock price $p$. The definition of the preferences is the same as in the VCG auction, apart from two changes. First, one has to replace the VCG strategy $S_i$ with the CCA strategy $(x_i, \{S^p_i\}_p)$. Second, the dynamic aspect of the CCA needs to be taken care of. A strategy $(\hat{x}_i, \{\hat{S}^p_i\}_p)$ is weakly preferred to another $(x_i, \{S^p_i\}_p)$ if for any history of the clock phase the continuation strategy is weakly preferred. This has two new implications relative to the definition for raising rivals’ cost given for the VCG mechanism. First, if the clock ended at price $\hat{p}$, the supplementary bidding function $\hat{S}^p_i$ is weakly preferred to $S^p_i$ if the minimum value of the implemented allocation with respect to the posterior $A(\hat{p})$ is larger. The difference with the VCG mechanism is that we use the posterior $A(\hat{p})$ rather than the prior belief $A$. Second, if the clock has not ended at price $\hat{p}$, then the evaluation of whether the continuation strategy $(\hat{x}_i|p \geq \hat{p}, \{\hat{S}^p_i\}_{p \geq \hat{p}})$ is weakly preferred to $(x_i|p \geq \hat{p}, \{S^p_i\}_{p \geq \hat{p}})$ again uses the posterior $A(\hat{p})$ and not the prior.

The equilibrium notion we use is a refinement of the standard ex post

\(^{12}\)Alternatively, one could also define the spite motive in terms of the expected sum of VCG prices, or in terms of maximizing the sum of VCG prices for every type profile. Our weaker and less risky formulation of the spite motive, is easier to work with and our results for the CCA do not depend on the specific formulation of the spite motive.
equilibrium notion applied to the first dimension of the preferences. That is we only consider ex post equilibria that are such that given the prior beliefs and the strategies of the others no bidder prefers to use a different strategy as defined above, including the preference to raise rivals’ cost. Clearly, we cannot use the notion of ex post equilibrium using the full preferences as in equilibrium we must allow for the fact that knowing the types of their competitors ex post, bidders may want to have changed their rivals’ cost raising bids.

3 The VCG mechanism

To better understand the role of the raising rivals’ cost motive, we first analyze the VCG mechanism under standard preferences using iterative elimination of weakly dominated strategies (IEDS). The first result shows that under standard preferences the outcome of IEDS is always efficient, but that the payments are undetermined and depend on the way IEDS is implemented. Truthful bidding is one of the strategies that survives IEDS, but, depending on the order of elimination, other strategies may survive IEDS as well. Bidders have to bid true marginal values on possible efficient shares in the interval \([x_i, \bar{x}_i]\) in order to get the efficient share. Outside the interval \([x_i, \bar{x}_i]\), bidders may bid differently as depending on the order of elimination bids on these shares may not be pivotal. As for weaker bidders, it is always the case that \(\bar{x}_i < 1\) these bidders have a range of shares for which the bid is undetermined by IEDS and the choice of these bids determines how much competitors have to pay. Accordingly, the payments in the VCG mechanism may well differ from the payments under truthful bidding.

**Proposition 1.** In the VCG mechanism with standard preferences, any strategy profile that survives any process of IEDS implements the efficient allocation. The VCG payments depend, however, on the order in which weakly dominated strategies are eliminated and on the choice of strategy profile that survives IEDS.

The elimination of weakly dominated strategies does lead to efficient outcomes, but the auction revenue is undetermined. The raising rivals’
cost motive may be viewed as a way to resolve the indeterminacy related to payments in an alternative way to imposing that bidders play their weakly dominant strategy. Under spiteful preferences truthful bidding is not an equilibrium in the VCG mechanism. To see this, suppose other bidders bid truthfully and consider a weak enough bidder with type \( a_i \) for whom the maximal efficient share \( \pi_i < 1 \). Without lexicographic preferences, bidder \( i \) is indifferent between some bids on \( (\pi_i, 1] \). A lexicographic bidder knows, however, that he can increase the price other bidders have to pay. The easiest way to do so is to increase the bid \( S_i(1) \) on the full supply as much as possible under the constraint that it is not winning.\(^{13}\) He never wins the full supply in an efficient equilibrium if for all \( a_{-i} \in A \)

\[
S_i(1) \leq S_i(x_i^*(a_i, a_{-i})) + \sum_{j \neq i} S_j(x^*(a_j, a_{-j})).
\]

The right-hand side of (2) depends on the types of the other bidders and is minimized if all other bidders have the lowest possible type \( a_i \). Hence, given our formulation of the spite motive bidder \( i \) wants to set the bid on 1 equal to the minimal value of the efficient allocation given bidder \( i \)'s type. If all bid true utility on \( [\pi_i, \pi_i] \), then the optimal bid is \( S_i(1) = V_i(a_i) \). Thus, bidders can use their private information and their knowledge about the lowest possible type of bidders to raise the bid on the full supply. Types that can win everything in an efficient equilibrium maximize rivals’ payment by bidding truthfully, in which case \( S_i(1) = U_i(1) = V_i(a_i) \).

The next proposition determines an efficient equilibrium under lexicographic preferences where bidders bid truthfully on all possible shares, apart from 1 if the type is low.\(^{14}\) The equilibrium strategies are increasing in \( x \), but not necessarily continuous at 1.

**Proposition 2.** The strategy profile in which bidder \( i = 1, \ldots, n \) with type

\(^{13}\)He could also increase his bid on other \( x \in (\pi_i, 1) \), but this does not create any benefit.

\(^{14}\)This is not the only equilibrium when bidders have lexicographic preferences. It is clear that bidders never want to bid above value on possible efficient shares. To protect themselves against others raising their price, bidders may however reduce their own bids on the domain of possibly efficient allocations without affecting their marginal bids.
forms an equilibrium of the VCG auction. There is a process of IEDS such that this strategy profile is iteratively undominated. This equilibrium is consistent with lexicographic preferences for raising rivals’ costs.

Note that the equilibrium specified in the proposition is not the unique equilibrium as, with more than two bidders, bidders may find ways to coordinate their strategies on more complicated schemes to raise rivals’ cost, for example by raising their bids on half of the supply, relying on others to do their share in raising rivals’ cost. These coordinated strategies may raise rivals’ cost beyond what is achieved in Proposition 2. However, given that in the strategy profile in (3) all bidders bid true utility on all shares smaller than 1, individually bidders cannot further raise the sum of VCG prices without running the risk of winning as the other bidders’ type profile may be such that the value of the efficient allocation is minimal. Hence, the strategy profile is an equilibrium under lexicographic preferences to raise rivals’ cost. Note that the strategy profile in (3) implements the efficient allocation and survives the IEDS of the proof of Proposition 1.

It is also important to note that all types \( a_i > a \) make positive surplus. This is because bidders do not want to risk winning the full supply and therefore are restricted in raising rival’s cost by the lowest possible efficient value \( V_i(a) \). If bidders would know their rivals’ types, they would fully expropriate them in any equilibrium where bidders bid valuation on the possibly efficient shares \([x_i, \bar{x}_i]\). Thus, in the VCG mechanism, bidders benefit from rivals being uncertain about their type.

4 Efficiency and Information Revelation in the CCA

In this section we present our first main result by discussing the fundamental trade-off between efficiency and information revelation in the clock phase. The clock phase can reveal information on competitors’ types through the price at which the clock phase ends. In particular, along the equilibrium path the clock may end at different prices for different type
profiles allowing bidders to learn competitors’ types.

The following function helps analyzing equilibrium information revelation. Let \( \tau : A \to \mathbb{R}_+ \) be a function that for given equilibrium strategies assigns to every type profile the final equilibrium clock price, i.e. \( \tau(a) = \inf\{p : \sum_i x_i(a_i, p) \leq 1\} \). There are two extreme cases in terms of information revelation. First, no information is revealed if the clock ends at the same price for all type profiles. In terms of the function \( \tau \), there exists a price \( p \) such that \( \tau(A) = \{p\} \). We call an equilibrium in which this is the case clock-pooling, since no bidder learns anything about other bidders’ types. We will provide an example of a clock-pooling equilibrium in the next section. Another extreme case is a clock-separating equilibrium which is defined as an equilibrium where the function \( \tau(a) \) is non-decreasing in \( a_i \) for all \( i \) and strictly increasing in \( a_j \) for all bidders \( j \) who win a positive share \( x_j < 1 \).

In a clock-separating equilibrium the final clock price conveys detailed information about the competitors’ types. A prominent example of a clock-separating equilibrium is truthful bidding. The following proposition shows that there is no efficient clock-separating equilibrium in the CCA. Truthful bidding is therefore not an equilibrium. In addition, if one interprets a clock-separating equilibrium as a formal definition of the notion of price and package discovery, mentioned in the introduction (see, e.g., Ausubel et al., 2006), then it follows that under a preference to raise rivals’ cost the CCA cannot deliver its two main objectives: efficiency and price and package discovery.

**Proposition 3.** There does not exist an efficient clock-separating equilibrium in the CCA.

In a clock-separating equilibrium all bidders must bid truthfully in the clock phase for all prices \( p \in [u_i(\bar{\pi}_i), u_i(\bar{\pi}_i)] \). However, weak bidders have an incentive to expand demand until at least a price \( u_i(\bar{\pi}_i) + \varepsilon \), for a fixed \( \varepsilon > 0 \), in the interior of this interval and then drop demand discontinuously to demand truthfully from then on. They can “correct” their deviation in the supplementary round by generating “missing” truthful bids in the

\[15\] The restriction \( x_j < 1 \) is needed for the following reason. Suppose \( a \in A \) is such that bidder 1 wins 1 in the efficient allocation. If bidders bid truthfully, for example, the clock would end by all bidders \( i > 1 \) dropping out truthfully. In this case the clock provides a lower bound for \( a_1 \), but the final clock price is not increasing in \( a_1 \).
interval $[x_i, \tau_i]$. Their competitors will not observe the deviation and simply (mistakingly) infer from the final clock price that the deviating bidder is a stronger type than his true type. As the competitors’ clock demands do not change, they will continue expressing the same supplementary bids so that the deviating player can continue acquiring the efficient share at the same price as without the deviation. The deviation weakens, however, the constraints imposed by the activity rules and allow the deviating bidder to raise rivals’ cost beyond what he could do if he would not have deviated. In the next section, we will show that this type of demand expansion can be part of equilibrium strategies that result in an efficient outcome.

5 Demand expansion and efficient equilibria

Efficient equilibria may well exist if the clock phase is not fully revealing. We illustrate this by means of two examples using the set-up of Levin and Skrzypacz (2016). The first example is a clock-pooling equilibrium as defined in the previous section. The second example presents a semi-separating equilibrium indicating that some, but not full, information revelation may well be consistent with efficiency. Both examples have bidders expand demand in the beginning of the clock phase and bidding truthfully afterwards. The examples are also important as they show that even if bidders anticipate their competitors engage in raising their cost in the supplementary phase, they do not have to engage in demand reduction strategies in the clock phase. This is in contrast to Levin and Skrzypacz (2016) who restrict themselves to linear proxy strategies and show that bidders will engage in demand reduction in the clock phase, assuming (against the auction rules) that a demand reduction strategy in the clock phase does not affect the ability to raise rivals’ cost. We conclude with a general result stating that in any efficient equilibrium of the CCA weaker bidders expand demand when they have a preference for raising rivals’ cost.

5.1 An efficient clock-pooling equilibrium

A clock-pooling equilibrium has the following structure. The clock phase starts with both bidders demanding the full supply. Demand is kept at this level for prices lower than $\tilde{p}$. This demand expansion weakens the
constraints of the activity rule and is necessary for being able to raise rival’s cost. Bidders start demanding truthfully at a certain price \( \tilde{p} \). The clock stops immediately at \( \tilde{p} \), because the price is so high that all types’ truthful demand \( \tilde{x}_i = \frac{a_i - \tilde{p}}{b} \) is smaller than half of the full supply. Formally, bidders use the clock demand function

\[
x_i(p) = \begin{cases} 
1 & \text{if } p < \tilde{p} \\
\max \left\{ \frac{a_i - p}{b}, 0 \right\} & \text{if } p \geq \tilde{p}.
\end{cases}
\] (4)

In the supplementary phase, bidders bid true utilities on possibly efficient shares and maximally raise the bid on the full supply such that it does not become winning and such that it satisfies the final and relative cap rule. The bids on \([0, \tilde{x}_i)\) and \((\tilde{x}_i, 1)\) are irrelevant as all bidders know they will not win shares in this interval and that bids on these shares do not determine the CCA prices. Hence, one can specify that bidders bid \( U_i(x) \) on all \( x < 1 \). The only thing that is left to be determined is how they can raise rivals’ cost as much as possible without running the risk of winning the full supply, while satisfying the activity rule. Whatever they bid in the supplementary round on their last clock round package \( \tilde{x}_i \), the final and the relative cap imply they can maximally bid \( \tilde{p}(1 - \tilde{x}_i) \) more on the full supply. However, if their bid on the full supply is more than \( S_i(\pi_i) - S_i(\tilde{x}_i) + S(1 - \pi_i) \) larger than their bid on \( \tilde{x}_i \) they run the risk of winning the full supply if the rival bidder’s type is low. Considering bidders bid value on \([\tilde{x}_i, \pi_i]\) and maximally raise rival’s cost without the risk of winning the full supply, bidders want to bid \( S_i(1) = \min \{ U_i(\pi_i) + U(1 - \pi_i), U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) \} \).\(^{16}\)

Figure 1 illustrates the clock-pooling equilibrium. The clock demand function on the left is such that 1 is demanded for prices \( p < \tilde{p} \). At this price, bidder \( i \) starts bidding truthfully. Since both bidders demand less than 1/2 at \( \tilde{p} \), the clock ends at this price. The plot on the right shows the supplementary bidding function and some constraints of the activity rule. To start with the constraints, the bid \( S_i(1) \) must be at least \( \tilde{p} \), and

\(^{16}\)Note that this strategy profile is discontinuous at \( x = 1 \). Under a lexicographic preference for raising rivals’ cost, equilibrium strategies are typically discontinuous at some point. Discontinuity is not so much an issue in real-world auctions as bidders typically have to bid on a discrete set of options and our model with a continuous supply is just a convenient simplification.
the bid \( S_i(\tilde{x}_i) \) must be at least \( \hat{p}\tilde{x}_i \). These constraints are depicted by the open dots. The bid on \( \hat{x}_i \) is raised to \( U_i(\hat{x}_i) \). By the final and the relative cap, \( S_i(x_i) \leq U_i(\hat{x}_i) + \hat{p}(x_i - x_i) \) for \( x_i \neq \hat{x}_i \). This constraint is depicted as the dashed line. Finally, the bid on 1 is chosen such that it satisfies the activity rule and it does not become winning. In the plot this is shown as \( \hat{p} < U_i(\pi_i) + U(1 - \pi_i) < U_i(\hat{x}_i) + \hat{p}(1 - x_i) \).

To see that these strategies may indeed form an efficient equilibrium, we first need to consider some restrictions that \( \hat{p} \) has to satisfy. Efficiency requires that all types are active in the last clock round and the price is such that all types are able to bid their marginal values in the supplementary round. This is the case if the final clock price is sufficiently high, i.e. if \( \hat{p} \geq \bar{a} - \frac{b}{2} \) (the marginal utility of the highest type at his smallest efficient share, which equals \( \frac{1}{2} \)). This price allows all bidders to bid their true marginal values in the relevant demand range \([\bar{x}_i, x_i]\) in the supplementary round. On the other hand, the final clock price \( \hat{p} \) cannot be too high either as otherwise some (weaker) bidders will bid above their true utility level giving the competitor a possibility to further raise their payments, yielding negative surplus. Thus, it must be that the final clock price \( \hat{p} \) the weakest possible type still derives positive utility if he acquires his highest possible efficient share, i.e., \( U(1/2) \geq \hat{p}/2 \), which is equivalent to \( \bar{a} - \frac{b}{2} \geq \hat{p} \).

Given the behavior in the clock phase it is clear that the bidding be-
behavior in the supplementary round is optimal. It is also clear that bidders do not have an incentive to lower their demand at \( p < \tilde{p} \) as it will not end the clock phase earlier (unless a bidder drops demand to 0) and it only potentially constrains the bidder in raising rival’s cost in the supplementary round (or making it impossible to get an efficient share). It remains to be seen whether bidders would like to continue the clock phase by not bidding truthfully at \( \tilde{p} \). The only reason why some bidders may want to do so is that in the candidate equilibrium constructed so far, they are not able to fully raise their rivals’ cost due to the activity rule, i.e. \( U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) < U_i(\bar{x}_i) + U(1 - \bar{x}_i) \). It turns out that there is a cutoff type \( \hat{a}(\tilde{p}) = \tilde{p}(2 + \sqrt{2}) - a(1 + \sqrt{2}) + b \) such that the last inequality is true if and only if \( \hat{a}(\tilde{p}) < a_i \). If \( \bar{a} \leq \hat{a}(\tilde{p}) \), then no bidder has an incentive to expand demand at prices higher than \( \tilde{p} \). This is the case when \( \tilde{p} \) is larger than \( \frac{\bar{a} + a(1 + \sqrt{2}) - b}{2 + \sqrt{2}} \).

Note that in these clock-pooling equilibria the clock phase always ends with excess supply, that the final equilibrium allocation is always ex post efficient and that bidders are able to fully raise their rivals’ cost without knowing the type of their rival. This type of efficient equilibrium does not exist if the uncertainty concerning bidders’ types, measured by \( \bar{a} - a \), is too large as in that case one cannot find a final clock price that satisfies the required properties. In particular, \( \bar{a} - \frac{b}{2} \leq \tilde{p} \leq a - \frac{b}{4} \) requires \( \bar{a} - a \leq \frac{b}{4} \).

### 5.2 An efficient clock-semi-separating equilibrium

In our next example, we show that some information revelation during the clock phase may arise in an efficient equilibrium. In the semi-separating equilibrium we analyze here, the clock-pooling equilibrium is adapted as follows. Like before, bidders demand the full supply at prices lower than \( \tilde{p} \leq a - \frac{b}{4} \). At \( \tilde{p} \) bidders start demanding truthfully. The clock ends at \( \tilde{p} \):

\[ U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) < U_i(\bar{x}_i) + U(1 - \bar{x}_i) \]

The working paper version (Janssen and Kasberger, 2016) develops further details. Alternatively, one can specify beliefs and behavior such that further demand expansion leads to a lower primary expected utility. Suppose the clock ends at \( p > \tilde{p} \). The equilibrium belief is that it was type \( \bar{x} \) who deviated. The equilibrium strategy specifies \( S^p_i(x) = U_i(x) \) for \( x < 1 \) and \( S^p_i(1) = \max\{U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i), V_i(\bar{a})\} \) for all \( a_i \). The final allocation is efficient, but the CCA price is strictly lower in exception if the clock ends at \( \tilde{p} \) than at \( p > \tilde{p} \).

\[ \text{Like in the clock-pooling equilibrium this condition guarantees that the lowest type makes nonnegative surplus in equilibrium.} \]
\( \hat{p} \) if both bidders are sufficiently weak, i.e. if \( a_i + a_j \leq 2\hat{p} + b \). Define \( \bar{a}(\tilde{p}) = 2\hat{p} + b - \pi \) to be the highest type for which the clock always ends at \( \hat{p} \). The observed clock phase behavior of these types is identical to what we encountered before. Types lower than \( \bar{a} \) never reveal any private information in the clock. For other types (and here is the difference with the clock-pooling equilibrium), the clock will not end if the competitor is of a sufficiently strong type. In this case, bidders continue to bid truthfully at prices larger than \( \hat{p} \), i.e., they use the clock demand function (4), and update their prior about the other bidder. As the clock proceeds after \( \hat{p} \), bidders gradually learn their competitor’s type as the lower bound of the belief concerning the competitor’s type is increasing in \( p \). Eventually, the clock will end with market clearing at \( p \) and bidder \( i \) knows the type of the other bidder to be \( 2\hat{p} + b - a_i \).

In the supplementary phase, if the clock ends at \( \hat{p} \), as in the clock-pooling equilibrium, bidders bid \( U_i(x) \) on all \( x < 1 \) and maximally raise rival’s cost without risking to win the full supply, so that \( S_i(1) = \min\{U_i(\pi_i) + U(1 - \pi_i), U_i(\hat{x}_i) + \hat{p}(1 - \hat{x}_i)\} \). Defining \( \hat{a}(\hat{p}) \) as the type for which \( U_i(\pi_i) + U(1 - \pi_i) = U_i(\hat{x}_i) + \hat{p}(1 - \hat{x}_i) \) it is not difficult to see that this minimum is reached at \( U_i(\pi_i) + U(1 - \pi_i) \) if, and only if, bidders have type \( a_i < \hat{a}(\hat{p}) \).

If the clock ends with market clearing at \( p > \hat{p} \), all types submit

\[
S^p_i(x) = \begin{cases} 
U_i(x) & \text{if } x \in [0, 1) \\
U_i(\hat{x}_i) + \hat{p}(1 - \hat{x}_i) & \text{if } x = 1
\end{cases}
\]

(5)

All types fully raise the bid on the full supply, since the final allocation is determined by the clock phase bidding (given market clearing at the final clock price). In the interior they maximally raise their (marginal) bids as this allows them to raise the bid on the full supply.

The proposed strategy profile is part of an equilibrium if \( \hat{a}(\hat{p}) \geq \hat{a}(\hat{p}) \). In this case, all types \( a_i > \hat{a}(\hat{p}) \) for which the clock phase possibly continues at prices \( p > \hat{p} \) cannot raise their supplementary bid on the full supply if the clock stops at \( \hat{p} \) without running the risk of sub-optimally winning the full supply. If, however, \( \hat{a}(\hat{p}) > \hat{a}(\hat{p}) \), then there exist types \( a_i \in [\hat{a}(\hat{p}), \hat{a}(\hat{p})] \) for which the clock phase does not necessarily stop at \( \hat{p} \) and their bid on the full supply depends on whether the clock stopped at \( \hat{p} \) or not. If the clock ended
at \( \tilde{p} \) they will bid \( U_i(\pi_i) + U_i(1 - \pi_i) \) as they do not want to risk winning the full supply, which happens if the competitor’s type is close to \( \bar{a} \). If the clock ends at a higher price, due to market clearing, they can safely bid \( S_i(1) = U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) \). Thus, the fact that the clock did not stop at \( \tilde{p} \) makes that for any fixed type in the interval \((\tilde{a}, \bar{a})\) their bid on the full supply jumps discretely by \( U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) \). Noting this, it is profitable for some types higher than \( \tilde{a}(\tilde{p}) \) to reduce demand at \( \tilde{p} \) to end the clock for sure. To guarantee that an efficient clock-semi-separating equilibrium exists it suffices that \( \tilde{a}(\tilde{p}) \geq \hat{a}(\tilde{p}) \), which together with \( \tilde{p} \leq \bar{a} - \frac{b}{4} \) holds if \( \bar{a} - \bar{a} < \frac{b}{4} \). If a semi-separating equilibrium exists it is efficient as all bidders bid their true marginal utilities in the supplementary phase and the bids are such that an interior solution arises.

5.3 Demand Expansion

In the two examples above, bidders expand demand in the first part of the clock phase. We will now show that this holds true more generally for any efficient equilibrium. In general, a bidder bids truthfully in the clock if at price \( p \) he demands an amount \( x_i \) such that \( u_i(x_i) = p \). We will say that a bidder expands demand in the clock phase if there are clock prices such that the bidder demands an amount \( x_i \) such that \( u_i(x_i) < p \). As marginal utilities are decreasing in \( x_i \), it is clear that this inequality can only hold if bidders demand more than their truthful demand.

The next proposition holds for the general model described in Section 2 with \( n \) bidders and general utility functions. The result states that in any efficient equilibrium weak bidders expand demand in the early stages of the clock phase.

**Proposition 4.** In any efficient equilibrium of the CCA there is a price

\[ \frac{\pi + a - b}{2} < \tilde{p} < \bar{a} - \frac{b}{4} \]

This condition also implies that one can find prices \( \frac{\pi + a - b}{2} < \tilde{p} < \bar{a} - \frac{b}{4} \). Along the equilibrium path it should be the case that even for the highest possible type \( \bar{\pi} \) the clock phase may possibly stop at \( \tilde{p} \). This requires that \( \tilde{p} > \frac{\pi + a - b}{2} \). The reason is that if a bidder with type \( \bar{\pi} \) knows that the clock will not stop even if he bids truthfully, he does not have an incentive to reduce demand and instead prefers to continue bidding on the full supply. As \( \tilde{a}(\tilde{p}) < \bar{\pi} \) the candidate equilibrium strategy implies he is restrained raising the rival’s cost if the clock stops at \( \tilde{p} \) and continuing bidding on the full supply would allow him to further raise the rival’s cost without affecting the final allocation (and the price he pays).
\[ \hat{p} = \tau(a) > u(a, 1/n) \] and an open set of (low) types that expand demand for prices \( p < \hat{p} \).

The idea behind this proposition is clear and similar to that of Proposition 3. In Section 2 we showed that under the general conditions we imposed, \( u(a_i, x_i) \) is increasing in \( a_i \). This implies that if, for some bidder types, the clock phase would end at a price smaller than \( u(a, 1/n) \), then bidders are restricted in their supplementary bids in a way that the outcome cannot be efficient for these types. Thus, if an efficient equilibrium is played, bidders know that the clock phase cannot end at a price smaller than \( u(a, 1/n) \). A deviation where a bidder would expand demand at prices smaller than \( u(a, 1/n) \), while keeping the rest of the clock phase bidding the same, would allow to raise rivals’ cost in the supplementary round in a way that does not risk winning these bids. Therefore, such a deviation increases bidders’ utility if they have a preference for raising rival’s cost.

6 No efficient equilibria with large uncertainty

So far we have shown that despite the fact that efficient clock-revealing equilibria do not exist, there may exist efficient equilibria in the CCA when bidders have a preference for raising rivals’ cost. The examples in the previous section use parameter values where the uncertainty concerning the final allocations, measured by \( \overline{\pi} - a \), is relatively small. In this section we will consider auctions where the uncertainty concerning the final allocations is large and show for the case of two bidders that the CCA does not have efficient equilibria.

**Proposition 5.** There is no symmetric and efficient equilibrium when there are two bidders and the ex ante uncertainty about the final allocation is high, i.e. \( u(\overline{\pi}, 1) > u(a, 0) \).

To prove this result, we introduce the following notation. Denote by \( \rho(a_i) \) the lowest clock price at which, along the equilibrium path, the clock phase may end for type \( a_i \). Formally, if the demand in the clock phase is non-decreasing in a bidder’s type, then \( \rho(a_i) = \tau(a_i, a_{-i}) \), where the function \( \tau(a_i, a_{-i}) \) is introduced in Section 4. In the efficient clock-pooling and clock-semi-separating equilibria presented in the previous section, we have
that $\rho(a_i) = \tau(a_i, a) = \tilde{\rho}$ for all types $a_i \in [a, \bar{a}]$, and thus, the image of $\rho$ singleton. In Janssen and Kasberger (2016) we presented an example of an efficient equilibrium where the image of $\rho$ is a pair of two prices.

The proof of the proposition first shows that if the uncertainty is large, there do not exist efficient equilibria where the image of the function $\rho$ is a singleton, like the clock-pooling and clock-semi-separating equilibria of the previous section, or a pair. The reason is simple. If the uncertainty is large, ex post efficiency requires that strong types continue bidding on the full supply for relatively large clock prices, while at these high clock prices weak types have to express bids above their utility in order to remain active in the clock phase. Stronger types will then have an incentive to raise the prices of the weak types so much that they make losses. Next, the proof establishes that independent of how large the uncertainty is, there do not exist efficient equilibria where the image of $\rho$ has more than two elements. The argument here is that if $\rho$ would have more than two elements, then the lowest possible types can condition their supplementary bidding function on the final clock price in such a way that the strongest types pay more than moderate types. From an efficiency point of view the strongest would need to continue the clock phase at prices where the moderate types would like to stop. However, individually a stronger bidder is better off reducing demand and pool with weaker bidders to prevent weak types extracting more of their surplus. This demand reduction is inconsistent with ex post efficiency, however.

The result is stated for the case of two bidders. The reason is that with more than two bidders, raising rivals’ cost may involve coordination with competitors. In this case, it is more difficult to prove that if an individual bidder expands demand in the clock phase he can strictly increase the prices others have to pay. It is clear it can never be worse to do so, but to prove that bidders would like to deviate, more is required. This issue does not arise with two bidders, as in that case it is an individual bidder’s bid on the full supply that determines the price competitors need to pay. Note also that the results holds true for utility functions that satisfy the general properties outlined in Section 2. For the model of Levin and Skrzyppacz (2016), one can show that efficient equilibria do not exist if a weaker condition holds, namely if $\bar{a} - a > \frac{b}{2}$ (Janssen and Kasberger (2016)).
7 Demand reduction and inefficient equilibria

Knowing that efficient equilibria do not exist if the uncertainty concerning a rival’s types is large, we now provide an example of an inefficient equilibrium using the set-up of Levin and Skrzypacz (2016). The inefficiency is due to demand reduction in the clock phase at relatively high prices as the strongest bidders know that if they would bid truthfully (or engage in demand expansion) at these prices, the clock phase may still continue and rival bidders will be able to raise their prices discontinuously. When both bidders have a preference for raising rivals’ cost, demand reduction takes on a different form, however, from what is discussed in Levin and Skrzypacz (2016). Demand reduction occurs at relatively high prices and bidders first still want to expand demand to be able to raise rivals’ cost as much as possible. Thus, bidders do not use linear demand functions in the clock phase as assumed by Levin and Skrzypacz (2016).

In the previous section we have introduced \( \rho(a_i) \) and defined it as the lowest price at which the clock phase may stop for type \( a_i \). In the equilibrium we construct here, there are two such prices \( \tilde{p}_1 \) and \( \tilde{p}_2 \) so that \( \rho(a_i) = \tilde{p}_1 \) for all types \( a_i \in [a, \tilde{a}_1) \) and \( \rho(a_i) = \tilde{p}_2 \) for all types \( a_i \in [\tilde{a}_1, \tilde{a}] \). Thus, the example also provides further detail how an equilibrium can be constructed where the image of \( \rho \) is a pair.

We define weak types as types in \( [a, \tilde{a}_1) \), where \( \tilde{a}_1 \) is the highest type for which the clock can possibly end if both bidders demand truthfully at \( \tilde{p}_1 \): \( \tilde{a}_1 = 2\tilde{p}_1 + b - a \). Weak bidders demand the full supply for all prices \( p < \tilde{p}_1 \) and demand truthfully at \( \tilde{p}_1 \). The price \( \tilde{p}_1 \) is chosen such that no weak type can further raise rivals’ cost if he learns that the rival is a strong bidder. This is the case if \( \hat{a}(\tilde{p}) \) as defined in Section 5, is not larger than \( a \) and we choose \( \tilde{p}_1 \) to be the largest price \( \hat{p} \) such that this is the case:

\[
\tilde{p}_1 = a - b/(2 + \sqrt{2}).
\]

It follows that \( \tilde{a}_1 = a + \frac{b\sqrt{2}}{2 + \sqrt{2}} = a + b(\sqrt{2} - 1) \), which is smaller than \( \tilde{a} \) if the uncertainty is large enough.

Strong types demand the full supply for all prices \( p < \tilde{p}_2 \), where \( \tilde{p}_2 > \tilde{p}_1 \). For simplicity, and to easily characterize the equilibrium structure, we

---

\(^{20}\)Note that there is a multiplicity of inefficient equilibria, as \( \tilde{p}_1 \) and \( \tilde{p}_2 \) can be chosen smaller than the values we chose here. We focus on precise values not to complicate the example too much. We refer to the working paper (Janssen and Kasberger, 2016) for further details.
choose $p^2 = a$. If $(\bar{a} - a)/b > 1/2$, there is a positive mass of types whose truthful demand at $\hat{p}^2 = a$, $(a_i - a)/b$, is more than $1/2$. We stipulate in the equilibrium we construct that these types reduce demand to $1/2$ in order to end the clock for sure at $\hat{p}^2$. To distinguish these types larger than $a + b/2$ from the strong bidders that drop demand truthfully to less than $1/2$ at $\hat{p}^2 = a$, we call these types “super-strong”. Thus, we have the clock demand function

$$x_i(p) = \begin{cases} 
1 & \text{if } p < \hat{p}^2 \\
\max\{\min\left\{\frac{a_i - p}{b}, \frac{1}{2}\right\}, 0\} & \text{if } p \geq \hat{p}^2
\end{cases}$$

(6)

of strong and super-strong types. Let $\hat{x}_i^j$, $j = 1, 2$, denote the truthful demand at price $\hat{p}^j$.

To finish the description of the clock phase, we still have to describe how weak types bid for prices $p$ such that $\hat{p}^1 < p < \hat{p}^2$. We specify that they bid according to true marginal values until they learn that the other bidder is strong. Given this strategy, a weak type $a_i$ learns that the rival bidder is a strong bidder if the clock phase is not over at a price $(\bar{a}^1 + a_i - b)/2$. Once they learn, their rival is strong, they keep demand at the level $\bar{x}_i^1 = x_i((\bar{a}^1 + a_i - b)/2)$ to be maximally able to raise their rival’s cost as long as $p \leq U_i(x_i)$. At this price they demand their truthful demand.

This description of clock phase behavior is summarized in Figure 2. Type profiles in the gray area are sufficiently weak so that the clock phase stops at price $\hat{p}^1$ and the it ends with excess supply with probability 1. If the bidders’ types are weak, but their profile is not in the gray area, then the clock phase stops at a price $p$ with $\hat{p}^1 < p < \hat{p}^2$. The interaction of these weak bidders is similar to what is described by the semi-separating equilibrium in Section 5.2. If the types $(a_i, a_j)$ are such that at least one bidder’s type $a_i > \bar{a}^1$, then the clock phase stops at price $\hat{p}^2$ and the interaction between bidders is similar to what is described by the clock-pooling equilibrium in Section 5.1.

We will now specify the behavior in the supplementary phase. If the clock ends at $\hat{p}^1$, then weak bidders submit the supplementary bidding

\[\text{21 As } (\bar{a}^1 + a_i - b)/2 < a \text{ all weak types learn this before the clock reaches price } \hat{p}^2.\]
Figure 2: Illustration of the inefficient equilibrium

function

\[ S^p_i(x) = \begin{cases} 
U_i(x) & \text{if } 0 \leq x < 1 \\
U_i(\hat{x}_i^1) + \hat{p}^1(1 - \hat{x}_i^1) & \text{if } x = 1.
\end{cases} \] (7)

Weak bidders bid true utility on all shares, but the full supply and maximally raise rivals’ bids by maximally raising their bid on 1. This is what is achieved by the bidding function (7).

If the clock phase ends at price \( \hat{p}_1 < \hat{p}^* < \hat{p}_2 \), it ends with market clearing and bidder \( i \) believes that the efficient share \( x_i^* \) has been implemented. Since they have submitted positive, truthful, bids for \( x \in [x_i^*, \hat{x}_i^1] \), they submit the bidding function

\[ S^{p^*}_i(x) = \begin{cases} 
U_i(x) & \text{if } x \in [0, \hat{x}_i^1] \\
U_i(\hat{x}_i^1) & \text{if } x \in (\hat{x}_i^1, 1) \\
U_i(\hat{x}_i^1) + \hat{p}^1(1 - \hat{x}_i^1) & \text{if } x = 1.
\end{cases} \] (8)

Bidders cannot further raise the competitor’s cost compared to the situation when the clock ends at \( \hat{p}_1 \), since the relative cap was already binding at \( \hat{p}^1 \). If the bidder is a weak type and the clock phase ends at \( \hat{p}_2 \) he will also bid according to (8).

Finally, if the clock ends at \( \hat{p}_2 \) strong bidders submit the following
supplementary bidding function

\[
S_i^{\tilde{p}^2} (x) = \begin{cases} 
U_i(x) & \text{if } 0 \leq x < 1 \\
U_i(\overline{x}_i) + U(1 - \overline{x}_i) & \text{if } x = 1
\end{cases}
\]  

while super-strong types bid according to the following strategy:

\[
S_i^{\tilde{p}^2} (x) = \begin{cases} 
0 & \text{if } x \in [0, 1/2) \\
U_i(\tilde{x}_i^2) + \tilde{p}^2(x - \tilde{x}_i^2) & \text{if } x \in [1/2, \tilde{x}_i^2) \\
U_i(x) & \text{if } x \in [\tilde{x}_i^2, 1) \\
U_i(\overline{x}_i) + U(1 - \overline{x}_i) & \text{if } x = 1
\end{cases}
\]

If the clock ends at \(\tilde{p}^2\) along the equilibrium path, strong bidders have received no information where their efficient share may lie and any share in the interval \([\underline{x}_i, \overline{x}_i]\) is possible. As it is well possible that their competitor is the weakest type possible, they will not use the ability to fully raise rivals’ cost as they risk winning the full supply at too high a price. In addition, super-strong bidders cannot bid true marginal values in the supplementary round that are higher than \(\tilde{p}^2\) as their clock phase behavior in combination with the local revealed preference rule does not allow them to do so. Thus, on shares smaller than \(\tilde{x}_i^2\), but larger than 1/2 they bid their maximum bids.

To complete the description of the equilibrium strategies, we specify some aspects of bidder behavior if the clock continues out-of-equilibrium at prices \(p > \tilde{p}^2\). What is important here is that bidders believe that the reason why the clock did not stop at \(\tilde{p}^2\) is that the super-strong bidders have deviated and demand more than 1/2. Given this belief the strong and super-strong bidders that did not deviate will respond to this deviation by adapting their bid on the full supply in the supplementary phase from what is specified in (10) to \(U_i(\tilde{x}_i^2) + \tilde{p}^2(x - \tilde{x}_i^2)\).

If these strategies are chosen, the allocation is \((1/2, 1/2)\) if both bidders are super-strong, as both bidders demanded 1/2 in the final clock round and the clock ended with market clearing. The final cap rule implies that this is the final allocation independent of the bidders’ true types. As this allocation is independent of bidders’ types, it is clear that this allocation is
inefficient. If a super-strong bidder $i$ meets another bidder $j$ and the sum of their truthful demands $(a_i - \hat{p}^2)/b + (a_j - \hat{p}^2)/b$ at the final clock price $\hat{p}^2$ is larger than 1, then the final allocation is $(1 - \hat{x}_j^2, \hat{x}_i^2)$, with $1 - \hat{x}_i^2 < \hat{x}_j^2$. It is clear that this allocation is also inefficient as it would be more efficient to give bidder $i$ more than $1 - \hat{x}_i^2$ and bidder $j$ less than $\hat{x}_j^2$. Clearly, the inefficiencies follow from the fact that super-strong bidders cannot express their true marginal utilities on all possibly efficient shares. If the bidders’ types are such that the above two cases do not arise, then the allocation in this equilibrium is efficient.

Intuitively, understanding why these strategies form an equilibrium requires a discussion of three crucial aspects:\footnote{A formal proof is given in the working paper of this paper (Janssen and Kasberger, 2016).} (i) why do superstrong bidders engage in demand reduction to end the clock phase, (ii) why do strong bidders not want to reduce demand to stop the clock phase earlier, and (iii) why do weak bidders not want to raise the cost of the strong bidders further? The understand the first aspect it is easiest to consider a case where $x(\bar{a}, \hat{p}^2)$ is slightly larger than $1/2$. By reducing demand to $1/2$ at $\hat{p}^2$ they do not loose much in terms of the direct utility they get from their final allocation compared to the situation where they would deviate and bid truthfully. Deviating gives rise to the clock phase continuing if the competitor is superstrong (which happens with positive probability) and in the case rivals will be able to raise the payment by a deviating bidder discontinuously. Thus, it is better to engage in demand reduction.

The second aspect follows from the fact that we have constructed $\hat{p}^1$ in such a way that $\hat{a}(\hat{p}^1) = \bar{a}$, i.e., in the equilibrium all weak types already raise rivals’ cost to the maximal extent possible and they cannot increase this cost further even if they learn the rival is a strong bidder. Strong bidders therefore have nothing to gain by reducing demand.

The third aspect requires an argument that weak bidders do not want to raise the cost of the strong bidders further by extending their demand on the full supply at prices $p > \hat{p}^1$ is more involved. Consider a type $a_i = a + b(\sqrt{2} - 1) - 2b\epsilon < \bar{a}^1$ for some $\epsilon > 0$. We will argue that there are some types of the rival bidder such that the only time type $a_i$ can get the efficient share is by bidding truthfully at $\hat{p}^1$. It is clear that the bidder does
not want to reduce demand as this will prevent him from always getting the efficient share at a price he wants to pay for it. Compare then the situation where he bids truthfully at $\tilde{p}^1$ and one where he expands demand. If he bids truthfully and the rival is of type $a_j = a + b\epsilon$, then their truthful demands at $\tilde{p}^1$ are

\[
\tilde{x}_i^1 = \frac{a + b(\sqrt{2} - 1) - 2b\epsilon - a - b\left(\frac{1}{\sqrt{2}} - 1\right)}{b} = \frac{1}{\sqrt{2}} - 2\epsilon
\]

\[
\tilde{x}_j^1 = \frac{a + be - a - b\left(\frac{1}{\sqrt{2}} - 1\right)}{b} = 1 - \frac{1}{\sqrt{2}} + \epsilon
\]

and under truthful demand the clock ends at $\tilde{p}^1$ with excess supply, i.e.

\[
\tilde{x}_i^1 + \tilde{x}_j^1 = \frac{1}{\sqrt{2}} - 2\epsilon + 1 - \frac{1}{\sqrt{2}} + \epsilon = 1 - \epsilon < 1.
\]

Importantly, note that the efficient share $x_j^* = \frac{a + be - a - b(\sqrt{2} - 1) + 2b\epsilon + b}{2b}$ for bidder $j$ is larger than his demand $\tilde{x}_j^1$. Given that the supplementary bidding function (7) applies in this case, the efficient allocation will be implemented. If, however, bidder $i$ expands demand so that the clock phase does not end at $\tilde{p}^1$, bidder $j$ believes that $a_i > a + b(\sqrt{2} - 1) - be$ and that his efficient share is smaller than $\tilde{x}_j^1$. Given the specification of the supplementary bidding function (8), he will only bid true marginal values on shares that are smaller than $\tilde{x}_j^1$, while the true efficient share is larger. Thus, the only time in the auction when $a_j$ submits true marginal values on efficient shares $x_j^*$ is when the clock ends at $\tilde{p}^1$. Consequently, if bidder $a_i$ does not drop demand to $\tilde{x}_i^1$, there is a positive probability he misses the chance of acquiring the efficient share and this reduces his expected surplus since

\[
U_i(x_i^1) + U_j(x_j^*) > U_i(1 - \tilde{x}_i^1) + U_j(\tilde{x}_j^1).
\]

Thus, as the other weak bidder bids truthfully at $\tilde{p}^1$, all weak bidders want to bid truthfully too, at least until they learn that the other bidder is not weak. This argument does not hold true for types $a_i \geq \tilde{a}^1$ as they have zero probability that the clock ends at $\tilde{p}^1$ under truthful bidding.

Finally, it is interesting to note that in this equilibrium types that are close to each other may pay very different amounts of money for very similar shares of the full supply. To see this, consider the difference in the CCA
price if a type \( \pi^1 - \varepsilon \) faces a type \( \pi^1 + \varepsilon \) when \( \varepsilon \) is arbitrarily small. The CCA price for the strong bidder \( \pi^1 + \varepsilon \) (facing a weak bidder \( \pi^1 - \varepsilon \)) is approximately equal to \( U_i(\pi_i) + U(1 - \pi_i) - U_i(\frac{1}{2}) \), where \( a_i = \pi^1 \), while the same price for the weak bidder \( \pi^1 - \varepsilon \) (facing a strong bidder \( \pi^1 + \varepsilon \)) is approximately equal to \( U_i(\hat{x}_i^1) + \hat{p}_1(1 - \hat{x}_i^1) - U_i(\frac{1}{2}) \). Thus, the relative price of the strong bidder \( \pi^1 + \varepsilon \) is approximately

\[
\frac{U_i(\pi_i) + U(1 - \pi_i) - U_i(\frac{1}{2})}{U_i(\hat{x}_i^1) + \hat{p}_1(1 - \hat{x}_i^1) - U_i(\frac{1}{2})} - 1 = 2 - \frac{4\sqrt{2}}{3} \approx 0.1143
\]

lower than what type \( \pi^1 - \varepsilon \) pays. In this equilibrium, the marginally weaker bidder is restricted to raise rivals’ cost due to his behavior in the clock phase, where the marginally stronger bidder does not face such restrictions.

8 Discussion and Conclusion

This paper provides a full equilibrium analysis of the CCA where the strategic interaction between the clock phase and the supplementary round is studied in an environment where bidders not only care about their own pay-off, but also (lexicographically) about how much rivals pay. We have argued that it is quite likely that bidders in real life auctions have an incentive to raise rivals’ cost and the lexicographic modeling of this preference is a very mild way of introducing this motive, while at the same time it provides a robustness check of the claims regarding the CCA.

We have two main results. First, there do not exist efficient equilibria of the CCA that fully reveal the type profile of the competitors in the clock phase. If rivals’ types are fully revealed, bidders can successfully raise their cost in the supplementary phase, providing an incentive to reduce demand and pretend to be a weaker type. This argument is related to the ratchet effect in the dynamic principal-agent literature where the agent does not want to reveal her type to a principal as the latter will use that information to extract more surplus from the agent in future interactions. In a CCA, the future interaction is represented by the supplementary phase and under raising rivals’ cost preferences a bidder both has an interest to extract surplus from their competitors (as the principal) and knows his surplus can be extracted by the others (as the agent).
Our second main result is that the CCA is inefficient if the uncertainty concerning final allocations is relatively large and there are two bidders. In an efficient equilibrium, bidders want to expand demand in the clock phase to be able to extract more surplus from their competitor in the supplementary phase. When the uncertainty is large, one cannot find final clock prices that are such that bidders are able to express their true marginal utilities on all possibly efficient shares. This creates the inefficiency. If the uncertainty concerning rivals’ types is small enough, efficient equilibria do exist, but they do not fully reveal information concerning rivals’ types. In this sense the clock phase does not fulfill its role as a price and allocation discovery process. As an example, we have characterized an equilibrium where no information is revealed during the clock phase and where bidders bid on the full supply before dropping demand truthfully. In this equilibrium, the clock phase ends with excess supply with positive probability.

It is difficult to assess whether or not real-world CCAs have been efficient as this would require knowing bidders’ utility functions. However, many of the equilibrium features of the CCAs we have highlighted show similarities to observed features of CCAs. Without pretending that there are no alternative explanations for these phenomena, we will give the following observations. First, after the 2013 auction the Austrian regulator RTR observed that during a large part of the clock phase, bidders’ demanded close to their full spectrum caps. This is in line with our examples on clock-pooling and clock-semi-separating equilibria and explained by our result on demand expansion in the clock phase. Second, the Austrian mobile network operator Telekom Austria (2013) indicates in a press release after the auction that the clock phase ended with excess supply in key spectrum bands. According to the Austrian regulator RTR this did not withhold the bidders to bid aggressively in the supplementary round (see, https://www.rtr.at/en/pr/PI28102013TK). This is also in line with our examples on clock-pooling and clock-semi-separating equilibria, where we argued that bidders create excess supply purposefully to obfuscate their type to prevent rivals to raise their cost.\footnote{The clock phase of the Canadian 700 MHz auction also ended with excess demand even though these units were allocated in the supplementary round.} Finally, the 2012 Swiss auction became known for bidders having to pay very different amounts for very
similar spectrum shares. Again, this is in line with our example of an inefficient equilibrium where very similar bidders pay very different amounts for almost identical spectrum shares.

Ausubel and Baranov (2015) have worked on alternative activity rules with the purpose of providing bidders with stricter incentives to bid according to their intrinsic preferences. They propose to replace the relative cap we used in this paper by GARP (the generalized axiom of revealed preference). We observe that in none of the three equilibria we constructed bidders violate GARP, and we conclude therefore that most of our results continue to hold if we would adopt the GARP activity rule.

Our paper extends the model by Levin and Skrzypacz (2016) in at least three important directions: bidders have general utility functions, many results consider more than two bidders, and the efficient allocation may involve some bidders leaving the auction empty-handed. Still, we have not dealt with complementaries in its full form (although we can accommodate relatively small complementaries in the analysis) and we have kept the format where only one unit of a perfectly divisible good is auctioned. Real-life telecommunications auctions are more difficult in that multiple units of different types of commodities are auctioned in one go. This creates additional complications as the relative prices between the different commodities should be taken into account. The extent to which our results generalize to these more complicated settings should be seen in follow-up research.

A Proofs

Proposition 1. In the VCG mechanism with standard preferences, any strategy profile that survives any process of IEDS implements the efficient allocation. The VCG payments depend, however, on the order in which weakly dominated strategies are eliminated and on the choice of strategy profile that survives IEDS.

Proof. First we show that with standard preferences, any strategy profile that survives any process of IEDS implements the efficient allocation. This proof has three parts. First, we show that bidding truthfully is an always
optimal strategy. Therefore, it survives any process of iteratively eliminating weakly dominated strategies (IEDS). Second, we argue that any bidder must be indifferent between any strategy that survived the IEDS and truthful bidding. In the third and final step we show inductively that only the efficient allocation can be implemented by strategies that survive IEDS. We will use the following notation. The set \( S_i \) is the set of strategies that survived IEDS for bidder with type \( a_i \). The set of iteratively undominated strategy profiles is denoted as \( S = S_1 \times S_2 \times \cdots \times S_n \).

First, bidding truthfully is an always optimal strategy, i.e., it is a best response against any strategy profile of other bidders \( S_{-i} \). To see this, let other bidders use \( S_{-i} \) and let \( \hat{x} \) denote the allocation implemented by the profile \((U_i, S_{-i})\). Note that \( \hat{x} \) is implemented only if \( U_i(\hat{x}_i) + \sum_{j \neq i} S_j(\hat{x}_j) \geq U_i(x_i) + \sum_{j \neq i} S_j(x_j) \) for all other feasible allocations \( x \). This inequality also says that the surplus of bidder \( i \) is at least as large under the allocation \( \hat{x} \) than under any other allocation. To see this, subtract the constant \( \sup_x \sum_{j \neq i} S_j(x_j) \) on both sides. Truthful bidding is always optimal and therefore \( U_i \in S_i \).

Second, bidder \( i \) must be indifferent between all \( S_i \in S_i \) and \( U_i \). For all \( S_i \in S_i \) it holds that for all other bidding functions \( T_i \) of bidder \( i \) and all bidding functions \( S_{-i} \in S_{-i} \) the surplus of \( S_i \) is at least as large as for \( T_i \) or strictly higher than for \( T_i \) for at least one \( S_{-i} \). Recall that the surplus from \( U_i \) is at least as large as from \( S_i \). As a result, the strategy \( S_i \) is iteratively not dominated if and only if for all \( S_{-i} \in S_{-i} \) the surplus of \( S_i \) and \( U_i \) is the same for all \( S_{-i} \in S_{-i} \).

We will now prove that any profile \( S \in S \) strictly implements the efficient allocation, i.e. \( \sum S_i(x^*_i) > \sum S_i(x_i) \) for all feasible allocations \( x \neq x^* \). We prove this assertion inductively. Let \( S = (S_1, U_2, \ldots, U_n) \in S \) be a strategy profile that implements \( \hat{x} \). When bidder 1 plays \( U_1 \) the efficient allocation \( x^* \) is strictly implemented. Note that bidder 1 is indifferent between \( S_1 \) and \( U_1 \), i.e.

\[
U_1(\hat{x}_1) + \sum_{j=2}^{n} U_j(\hat{x}_j) = U_1(x^*_1) + \sum_{j=2}^{n} U_j(x^*_j).
\]

Since \( x^* \) is the unique efficient allocation, \( \hat{x} = x^* \) must be true.
Suppose \((S_1, S_2, \ldots, S_k, U_{k+1}, \ldots, U_n)\) implements the efficient allocation, with \(k = 1, \ldots, n - 1\). Then also \((S_1, \ldots, S_{k+1}, U_{k+2}, \ldots, U_n)\) implements the efficient allocation. Let \(\hat{x}\) denote the allocation implemented by the profile \((S_1, \ldots, S_{k+1}, U_{k+2}, \ldots, U_n)\). Bidder \(k + 1\) must be indifferent between \(S_{k+1}\) and \(U_{k+1}\), that is,

\[
U_{k+1}(\hat{x}_{k+1}) + \sum_{j=1}^{k} S_j(\hat{x}_j) + \sum_{j=k+2}^{n} U_j(\hat{x}_j) = U_{k+1}(x^*_{k+1}) + \sum_{j=1}^{k} S_j(x^*_j) + \sum_{j=k+2}^{n} U_j(x^*_j),
\]

where the right-hand side denotes the surplus from bidding truthfully. By the induction hypothesis the allocation implemented by bidder \(k + 1\) bidding truthfully is the efficient allocation. The left-hand side is the surplus from bidding \(S_i\) and at the same time says that bidder \(k + 1\) bidding truthfully implements allocation \(\hat{x}\). Again by the induction hypothesis, \(\hat{x} = x^*\) must be true.

The proof that the VCG prices depend on the process of IEDS is constructive. We construct a sequence of eliminations that ends with a set of undominated strategies. Strategies in the set will have the desired properties. In order to show that a strategy is dominated, one needs to find an alternative strategy that yields weakly higher utility against all admissible strategy profiles of the other bidders and a strictly higher utility against some admissible strategy profiles. Above we have seen that bidding \(U_i\) is always optimal. In the subsequent three steps of iterative elimination we only have to construct a profile \(S_{-i}\) to show that the alternative of truthful bidding is strictly preferred.

Let \(B\) be the set of all bidding functions, i.e. the set of all \(S : [a, \bar{a}] \times [0, 1] \to \mathbb{R}_+\). Note that the optimality of a function depends on the type \(a_i\).

**Step 1:** Strategies \(S_i\) for which there exists \(\hat{x} < 1\) such that \(S_i(\hat{x}) > U_i(\hat{x})\) are dominated. Consider bidder \(j \neq i\). All other bidders \(k \neq j, i\) bid \(S_k(x) = 0\) for simplicity. Bidder \(j\) uses the bidding function

\[
S_j(x) = \begin{cases} 
\max_y S_i(y) + S_i(\hat{x}) & \text{for } x = 1 \\
\max_y S_i(y) & \text{for } x = 1 - \hat{x} \\
0 & \text{else.}
\end{cases}
\]
The bidding profile $S$ implements the allocation in which bidder $i$ wins $\bar{x}$ and bidder $j$ wins $1-\bar{x}$, since ties are broken in favor of interior allocations. Bidder $i$’s surplus is $U_i(\bar{x}) - S_i(\bar{x}) < 0$, whereas the surplus from bidding truthfully is non-negative. Remove these dominated strategies to obtain $S^1 \subset B$.

Step 2: Strategies are dominated that satisfy $S_i(1) > V_i(a)$. Note that for low types $V_i(a) > U_i(1)$, but for high types, with $\bar{x}_i = 1$, $U_i(1) = V_i(a)$. Bidders $k \neq i, j$ bid $S_k(x) = 0$ for all $x$ and bidder $j$ bids $S_j(x) = 0$ for $x < 1$ and $S_j(1) = S_i(1) - \varepsilon$, with $\varepsilon \in (0, S_i(1) - V_i(a))$. Bidder $i$ wins the full supply at a price higher than true utility. Truthful bidding is therefore strictly better against this strategy profile of other bidders. Remove these dominated strategies to get $S^2 \subset S^1$.

Step 3: Strategies are dominated for which there exists $\tilde{x} \in [x_i, \bar{x}_i]$ with $U_i(\tilde{x}_i) > S_i(\tilde{x})$. Let again bidders $k \neq j, i$ bid $S_k(x) = 0$ for all $x$. Let $x' \in \arg\max_y S_i(y)$. Let $\varepsilon \in (0, U_i(\tilde{x}) - S_i(\tilde{x}))$. In the case of $\tilde{x} < 1$, suppose bidder $j$ uses the bidding function $S_j(1) = S_i(x') + \varepsilon$, $S_j(1-\tilde{x}) = S_i(x') - S_i(\tilde{x})$ and $S_i(x) = 0$ for all other $x$. Under this bidding function, bidder $i$ wins nothing and gets zero surplus. If the bid on $\tilde{x}$ is raised to $U_i(\tilde{x})$, then bidder $i$ wins $\tilde{x}$ and gets strictly positive surplus. For $\tilde{x} = 1$, let $S_j(1) = S_i(1) + \varepsilon$ and 0 otherwise. Bidder $i$ wins nothing if the bid is below true utility level and the full supply if the bid equals utility. The set $S \subset S^2$ is obtained by eliminating these dominated strategies.

After the three steps of elimination, all remaining strategies implement the efficient allocation. To see this, let bidders use the admissible strategy profile $S \in S$. The value jointly expressed for the efficient allocation is higher than the value jointly expressed for any other feasible allocation $x$. Let $x_i < 1$ for all $i$. Then

$$\sum_{i=1}^{n} S_i(x_i) \leq \sum_{i=1}^{n} U_i(x_i) \leq \sum_{i=1}^{n} U_i(x_i^*) = \sum_{i=1}^{n} S_i(x_i^*),$$

where the first inequality follows from step 1, the second inequality from the definition of efficiency, and the last equality from steps 1, 2 and 3. For
an allocation such that there is an $i$ with $x_i = 1$ we have
\[ \sum_{j=1}^{n} S_j(x) = S_i(1) \leq \sum_{j=1}^{n} U_j(x_j^*) = \sum_{j=1}^{n} S_j(x_j^*), \]
where the first equality follows from step 1, the first inequality from step 2, the second inequality from the definition of efficiency, and the last equality from steps 1, 2 and 3. All strategy profiles in $\mathcal{S}$ implement the efficient allocation. There are no further dominated strategies, since any strategy that survives IEDS yields the same expected utility as bidding truthfully.

To see that the VCG prices depend on the chosen strategy profile, consider a bidder with $a_i$ sufficiently small so that $V_i(a) < U_i(1)$ and all other players having the lowest possible type $a$. The value of the efficient allocation is $V(a)$. Suppose bidder $i$ chooses $S_i \in \mathcal{S}_i$ with $S_i(x) = U_i(x)$ for $x < 1$ and $S_i(1) = V_i(a)$ and all other bidders play $S_j = U_j$. The following inequalities hold $S_i(1) = \sum_{i=1}^{n} U_i(x_i^*) > \max_x \sum_{j \neq i} S_j(x) = \max_x \sum_{j \neq i} U_j(x)$. Hence, the VCG price for bidder $j$ is $S_i(1) - \sum_{k \neq j} U_k(x_k^*)$. If the strategy profile was such that $S_i = U_i$, then the VCG price would be strictly less than that and equal to $\max_x \sum_{j \neq i} U_j(x) - \sum_{j \neq i} U_j(x_j^*)$. Note that the strict inequality and continuity imply that the difference in VCG prices holds for an open set of types. Similarly, if step 1 was such that we eliminate also $S_i(1) > U_i(1)$, then the first VCG would not be possible. \[ \blacksquare \]

**Proposition 3.** There does not exist an efficient clock-separating equilibrium in the CCA.

**Proof.** Suppose an efficient clock-separating equilibrium exists. Clock separation requires that demand is monotone in type, i.e. $a_i' \geq a_i$ implies $x_i(a_i', p) \geq x_i(a_i, p)$. Let $\mathcal{A}'$ be an open neighborhood of the type profile $a \in A$ so that for any $a \in \mathcal{A}'$ all bidders are winners in the efficient allocation. The equilibrium strategy profile must have the following properties.

First, the clock must end with market clearing almost surely. Suppose there is a positive probability, i.e. an open set of type profiles $\mathcal{A}'' \subseteq \mathcal{A}'$, that the clock ends with excess supply. The clock ends with excess supply only if a bidder uses a demand function with discrete downward jumps. Without loss of generality, let bidder 1 make a jump that ends the clock for type profile $a \in \mathcal{A}''$ at $\tau(a) = p$. Consider type profiles in which $a_{-1,2}$ is fixed, but
$a'_2$ is slightly smaller than $a_2$. Then the clock must end with excess supply at 
$\tau(a_1, a'_2, a_{-1.2})$ by bidder 2 making downward jumps, because $\tau$ is increasing 
in $a_2$. Fix $a'_2$ slightly smaller than $a_2$ such that $\tau(a_1, a'_2, a_{-1.2}) = p' < p$ and 
bidder 1’s demand with type $a_1$ has no jumps on $[p', p)$. For a given $a_{-i}$, the 
function $\tau_i(a_i) = \tau(a_i, a_{-i})$ is strictly increasing in $a_i$, and therefore almost 
everywhere continuous. Hence, there exists a type $a'_1$ slightly larger than $a_1$ 
such that $\tau(a'_1, a'_2, a_{-1.2}) = p''$ and $p' < p'' < p$. At $p''$ it is bidder 1’s discrete 
decrease that ends the clock. Since demand functions are monotone in type, 
it must be that type $a_1$ drops demand at $p''$, a contradiction. It cannot be 
that the clock ends with excess supply with positive probability.

Second, the relative cap is binding for relevant shares in $(x^*_i, 1]$ in any 
supplementary phase on the equilibrium path. Since the clock ends with 
market clearing almost surely and the equilibrium is efficient, demand in 
the final clock round must be the respective efficient shares. This follows 
from the definition of the final cap rule. If the relative cap was not binding, 
then as the clock continues one can increase supplementary bids on shares 
that determine other bidders’ CCA prices relative to the efficient share. 
As a consequence, the price paid by a rival bidder would be dependent on 
the final clock price. Demand reduction would then be a best response for 
some types.

Third, bidders need to demand truthfully for $p \in [u_i(x_i), u_i(x_i)]$. The 
clock ends with market clearing and the relative cap is binding. Suppose 
$u_i(x^*_i) < \tau_i(a)$, that is, the bidders did not bid truthfully in the final clock 
round, but every bidder demanded the efficient share. Then the marginal 
bid $s_i(x) > u_i(x)$ for $x > x^*_i$ and therefore the marginal CCA price is not 
such that bidders want to win the efficient share. This cannot be the case 
in an efficient equilibrium. As a result, every bidder bids $s_i(x) = u_i(x)$ for 
$x \in [x_i(p^*), x_i]$ for any final clock price $p \in [u_i(x_i), u_i(x_i)]$.

We now show that given these properties there is a profitable deviation 
from the clock-separating equilibrium strategy. This deviations that leads 
to the same expected utility in the first dimension of the preferences, but 
to strictly higher CCA prices for some possible final clock prices. Bidder 
i deviates by first expanding demand for some prices strictly above $u_i(x_i)$ 
and bids truthfully at some price $p > u_i(x_i)$ and from then on. If the clock 
ends at $p$, it almost surely ends with excess supply. Other bidders do not
see the deviation and believe that the clock ended with market clearing. They fully raise the supplementary bids to $s_j(x) = u_j(x)$ for $x \in [x_j(p), \pi_j]$. Hence, a suitable level of $S_i(x_i(p))$ and true marginal values $s_i(x) = u_i(x)$ on $[x_i(p), \pi_i]$ implement the efficient allocation. The CCA price for the deviating bidder is the same as under the initial strategy, since the CCA price is independent of the final clock price. The CCA to other bidders is not less than the ‘equilibrium’ prices if the clock ends at $p$. If the clock does not end at $p$, it will end at a higher clock prices with market clearing. The deviation weakened the constraints of the activity rule, hence the bids on $(\pi_i, 1]$ are larger than of the initial strategy and lead to a larger CCA price for other bidders.

**Proposition 4.** In any efficient equilibrium of the CCA there is a price $\tilde{p} = \tau(a) > u(a, 1/n)$ and an open set of (low) types that expand demand for prices $p < \tilde{p}$.

*Proof.* It is clear that in an efficient equilibrium, the clock phase cannot end at a price smaller than $u(a, 1/n)$. Otherwise, bidders are restricted by the relative cap and cannot bid true marginal utility on all possibly efficient shares. The proof of Proposition 3 shows that there is no efficient equilibrium in which $\tau(a) = u(a, 1/n)$. Hence, it must be the case that $\tau(a) > u(a, 1/n)$. This price is the lowest final clock price only if low types, that is, types in an open neighborhood of $a \in A$ expand demand. Moreover, without demand expansion bidders would not be able to fully raise rivals’ costs. □

**Proposition 5.** There is no symmetric and efficient equilibrium when there are two bidders and the ex ante uncertainty about the final allocation is high, i.e. $u(\pi, 1) > u(a, 0)$.

*Proof.* We first prove two properties any efficient and symmetric equilibrium needs to satisfy. Then we use these properties to show that if the uncertainty is large, i.e., $u(\pi, 1) > u(a, 0)$, no symmetric and efficient equilibrium exists where the image of the function $\rho$ is a singleton or a pair. Finally, we show that independent of the uncertainty there cannot be an efficient and symmetric equilibrium where the image of the function $\rho$ is neither a singleton nor a pair. Throughout the proof, we will say that
the relative cap is binding if the bid on 1 relative to the bids on efficient shares is the same in every supplementary phase, i.e. if, from bidder $j$’s perspective, $S_i^p(1) - S_i^p(x_i^*)$ only depends on $a_i$ but not on final clock price $p$.

The first property is that the relative cap must be binding in every supplementary phase along the equilibrium path if it is not the case that $\tau(a_i, a_j)$ is independent of $a_i$ and $a_j$, i.e., there are at least two distinct final clock prices possible. If this was not true, then a bidder could increase the price paid by the other bidder once it is known that the clock continues for prices above the smallest $\tau(a_i, a_j)$. It is clear that this uses the fact that $n = 2$, because with one rival, a bidder knows that how much the rival pays depends on his bids on 1 alone.

The second property is that (as the relative cap must be binding along the equilibrium path) bidders will expand demand in the clock phase to relax the relative cap if they can do so without affecting the equilibrium allocation. There are two instances in the clock where they can do so. The first instance is at the beginning of the clock, implying bidder $i$ with type $a_i$ demands 1 until the price is $\rho(a_i)$. The second instance is the following. Suppose bidder $i$ with type $a_i$ knows that for $a_j < a'$ the clock must end no later than $p' = \sup_{a_j < a'} \tau(a_i, a_j)$. Suppose that $\tau(a_i, a') > p'$ and the clock has not ended at $p'$. Then bidder $i$ can expand demand in the clock phase by keeping demand constant between $p'$ and $\tau(a_i, a')$.

We now prove that if $u(\bar{a}, 1) > u(\bar{a}, 0)$, it cannot be that the image of $\rho$ is a singleton or a pair. We will prove this as follows. Denote the two clock prices in the image of $\rho$ by $\hat{p}_1$ and $\hat{p}_2$ with $\hat{p}_1 < \hat{p}_2$. If we can prove that the price $\hat{p}_2$ has to be strictly smaller than $u(\bar{a}, 0)$, then it follows from $u(\bar{a}, 1) > u(\bar{a}, 0)$ that $\hat{p}_2 < \min\{u(\bar{a}, 1), u(\bar{a}, 0)\}$ and then in an efficient equilibrium it must be that the clock will necessarily continue beyond $\hat{p}_2$ for at least some types close to $\bar{a}$, which is in contradiction to the fact that the image of $\rho$ is a pair. So, the only thing we need to establish is that $\hat{p}_2$ has to be smaller than $u(\bar{a}, 0)$. Suppose to the contrary that $\hat{p}_2 \geq u(\bar{a}, 0)$. Then the lowest possible type has to expand demand between $\hat{p}_1$ and $\hat{p}_2$ (or has to engage in demand expansion until $\hat{p}_1 = \hat{p}_2$ (if the

\footnote{In Subsection 5.2 in the clock-semi-separating equilibrium the binding relative cap requirement manifests itself in $\hat{a} \leq \bar{a}$.}
image is a singleton), because otherwise the clock ends at a price below $\hat{p}^2$. As a consequence of the demand expansion, the lowest type needs to bid above utility for possible efficient shares (and also on 0). We will show that bidding above utility leads to negative primary surplus for some type combinations. Define $a'$ to be the lowest type for which $x_i = 1$, i.e. $u(a, 0) = u(a', 1)$. As $u(\bar{a}, 1) > u(a, 0)$ such a type $a'$ exists. Let $a_i$ be slightly smaller than $a'$. Type $a_i$ needs to demand truthfully at $\hat{p}^2$ in order to implement the efficient allocation. Suppose the relative cap binds for the bid $S_i(1) = S_i(\hat{x}_i^2) + \hat{p}^2(1 - \hat{x}^2)$, and bidding $S_i(x) = U_i(x) - U_i(\hat{x}_i^2) + S_i(\hat{x}_i^2)$ for $x \in [\hat{x}_i^2, 1)$ implements the efficient allocation. When the lowest type meets $a_i$, the efficient share is implemented and the lowest type’s surplus is

$$U(1 - \bar{x}_i) - S_i(\hat{x}_i^2) - \hat{p}^2(1 - \hat{x}^2) + U_i(x_i) - U_i(\hat{x}_i^2) + S_i(\hat{x}_i^2) < 0 \iff$$

$$U_i(\hat{x}_i^2) + \hat{p}^2(1 - \hat{x}^2) > U(1 - \bar{x}_i) + U_i(x_i).$$

The last inequality is true, because the right-hand side is just slightly larger than $U_i(1)$, but bidder $i$ expanded demand, so the left-hand side is discretely larger than $U_i(1)$. As a result, if the relative cap is binding and the efficient allocation is implemented, then the lowest type makes a loss in the efficient allocation. When a binding relative cap does not implement the efficient allocation, then the proposed strategy profile cannot be an efficient equilibrium, as the argument above shows. Thus, $\hat{p}^2 < u(a, 0)$ and the image of $\rho$ cannot be a pair.

The last substantial part of the proof is to show that independent of the uncertainty there cannot be an efficient and symmetric equilibrium where the image of the function $\rho$ is neither a singleton nor a pair. First, it is important to note that due to the arguments given in the proof of Proposition 3 it cannot be the case that the image of $\rho$ contains a range of prices $[\hat{p}^1, \hat{p}^1 + \varepsilon)$ for some $\varepsilon > 0$, where $\hat{p}^1$ is the smallest of the prices in the image of $\rho$. It remains to be shown that it cannot be the case that the image of $\rho$ has two final clock prices $\hat{p}^1$ and $\hat{p}^2$ as its lowest two prices and that the clock continues after $\hat{p}^2$ for some set of types $[\bar{a}, \bar{a}')$. If this were the case then it should be that there exists $\bar{a}'$ and $\bar{a}''$ with $a < \bar{a}' < \bar{a}'' \leq \bar{a}$ and $\bar{a}''$ being at most the lowest type for which the highest possible efficient share is 1, i.e. $x^*(a, a_j) > 0$ for all $a_j < \bar{a}'$, such that low types drop demand
at the same price, i.e., \( \rho([a, \bar{a}')) = \hat{p}_1 \) and higher types drop demand at a higher clock price and this clock price is such that the clock ends when meeting the lowest type, so \( \rho([\bar{a}', \bar{a}'']) = \hat{p}_2 \).

As \( \rho([a, \bar{a}')) = \hat{p}_1 \) it is clear that \( x(a, \hat{p}_1) < 1/2 \) so that there must exist some \( \delta > 0 \) such that if both types are in \([a, a + \delta] \) the clock phase will stop at \( \hat{p}_1 \). We will now show that all types in \([a, a + \delta] \) must demand truthfully. It is clear that a type \( a_i \in [a, a + \delta] \) cannot reduce demand at \( \hat{p}_1 \), i.e., \( x_i = \hat{x}_i, \) since it prevents bidding true marginal values on efficient shares, and it limits the possibility to raise rival’s costs as well. Let us then consider the case where their demand \( x_1 \) is in \([\hat{x}_1, 1/2] \). In any supplementary phase on the equilibrium path, bidder \( i \)'s bid on efficient shares must be the same relative to the bid on 1 and the relative cap must be binding, so bidder \( i \) bids \( S_i(x) = U_i(x) - U_i(x_1) + S_i(x_1) \) for \( x \in [x_i, \bar{x}_i] \) and on 1 raises the bid maximally to \( S_i(1) = S_i(x_1) + \hat{p}_1 (1 - x_1) \). The CCA price for an efficient share \( S_i(1) - S_i(x_1) = S_i(x_1) + \hat{p}_1 (1 - x_1) - U_i(x_1) + U_i(x_1) - S_i(x_1) \) is then maximized at \( x_1 = \hat{x}_1 \), since \( \hat{x}_1 \) satisfies the first order condition with respect to \( x_1 \); \( u_i(\hat{x}_1) = \hat{p}_1 \). Hence, all types in \([a, a + \delta] \) must demand truthfully at \( \hat{p}_1 \).

Given that weak bidders in \([a, a + \delta] \) must demand truthfully at \( \hat{p}_1 \), it is clear that in an efficient equilibrium these weak bidders cannot reduce demand at prices \( \hat{p}_1 < p < \hat{p}_2 \) to levels smaller than their truthful demand \( \hat{x}_i(p) \) as this will imply they are restricted to bid true marginal utility in the supplementary phase on possibly efficient shares.

We will now show that there cannot be an equilibrium where these weak types bid truthfully at all prices \( p \) with \( \hat{p}_1 < p < \hat{p}_2 \) as they will earlier learn that their competitor is of a type \( a_j > \bar{a}' \) and use this information to raise rivals’ cost. Truthful bidding leads to the supplementary bidding constraint \( s_i(x) \leq u_i(x) \). This constraint must be binding in equilibrium, since it maximally raises the CCA price of the competitor. Hence, types above \( \bar{a}' \) have no incentive to reduce demand at prices higher than \( \hat{p}_1 \), because they can get the efficient share at later final clock prices, too. These types expand demand and drop demand at \( \hat{p}_2 \). Thus, there must exist some \( \delta_1 > 0 \), such that types in \([a, a + \delta_1] \) learn that the other bidder’s type is larger than \( \bar{a}' \). They learn this at \( p = u(a_i, \bar{a}') \geq \hat{p}_1 \) and for small enough \( \delta_1 \) we have \( u(a_i, \bar{a}') < \hat{p}_2 \). From this price bidders with type
in \([a, a + \delta_1]\) keep demand constant until \(\hat{p}^2\). As \(\hat{p}^2 \leq u(a, 0)\) these weak types demand a positive amount and the clock will continue with positive probability. If it continues, they can raise the rival’s CCA price. In the supplementary round, this bidder will use different supplementary round bid functions, depending on whether the clock phase stopped at a price \(p \leq \hat{p}^2\) or whether the clock phase stopped at a price \(p > \hat{p}^2\). In particular, if the clock phase stopped at a price \(p \leq \hat{p}^2\) these types will bid true marginal values on shares \(\hat{x}_1^i \leq x \leq \hat{x}_1\) and raise the price the competitor pays to \(\hat{p}^1 (1 - \hat{x}_1^i) + U_i(x^*_i) - U_i(\hat{x}_2^i)\). On the other hand, if the clock phase stops at a price \(p > \hat{p}^2\) these types in \([a, a + \delta_1]\) will express true marginal values only on shares \(x < \hat{x}_2^i\). Hence, they can use the demand expansion before \(\hat{p}^2\) to raise the CCA price to \(U_i(x^*_i) - U_i(\hat{x}_2^i) + \hat{p}^2(\hat{x}_1^i - \hat{x}_2^i) + \hat{p}^1 (1 - \hat{x}_1)\). This will give types just above \(\bar{\pi}'\) an incentive to reduce their demand in the clock phase to imitate types just below \(\bar{\pi}'\).

Finally, suppose some types in \([a, a + \delta_2]\) keep demand constant at \(\hat{x}_1^i = x(a, i, \hat{p}^1)\) for prices \(p > \hat{p}^1\), i.e., they expand demand until some \(p' > \hat{p}^1\) and demand truthfully at \(p'\). A similar argument as in the preceding paragraph can be used to argue that these bidders will be able to raise rivals’ cost further if they learn the clock does not stop at price \(\hat{p}^2\) and that types just above \(\bar{\pi}'\) then have an incentive to reduce demand and imitate types just below \(\bar{\pi}'\).

References


