Final exam

1. Consider an auction with two buyers and independent private values uniformly distributed in the interval \([0, 1]\).
   (a) What is the expected revenue to the seller in a standard auction \textit{without} reserve price where a buyer with value \(x = 0\) has zero expected payment?
   (b) Instead, what would be the expected revenue to the seller in a second-price, sealed-bid auction \textit{with} reserve price \(r = 1/2\)?
   (c) Consider now the following selling mechanism which is commonly used to sell a house when a renter occupant has the right to match the offer of a potential buyer. Buyer 1 can submit an offer \(b \geq 1/2\), where 1/2 is the seller’s reserve price. After observing \(b\), buyer 2 has the choice to match this offer and win the object at price \(b\). If buyer 1 does not submit any offer, then buyer 2 can obtain the good at price 1/2, if she wants. Buyer 1 gets the good at price equal to her offer \(b\) if she submits an admissible offer that is not matched by buyer 2.
      i. If buyer 1 submits an admissible offer \(b\), when should buyer 2 with value \(x_2\) match this offer? What should buyer 2 do if buyer 1 does not submit any offer?
      ii. What is the optimal behavior of buyer 1 if her value \(x_1 < 1/2\)? What is her optimal offer \(b^*\) if \(x_1 \geq 1/2\), given the optimal reaction of buyer 2?
        \text{Hint:} Note that \(x - 2b < 0\) for \(b > 1/2\) since \(x \leq 1\).
      iii. What will buyers do in the equilibrium you just obtained and what is the expected revenue to the seller? Compare to the revenues of a standard auction without reserve and the second-price auction with \(r = 1/2\). Is this selling mechanism optimal? Is it efficient?

2. Advertising space is auctioned among two firms who are also competitors in the aftermarket. Winning the auction results in a profit increase of \(X_i\) to firm \(i\) coming from advertisement. If the competitor wins the ad, then firm \(i\) suffers a profit loss of \(X_i/2\), due to the loss in market share. Assume \(X_i \sim U[0, 1]\) and the value of profit increase is private information to each firm.
   (a) If the ad is sold using a first-price auction, what are the equilibrium bidding strategies? What is the expected revenue to the seller of the ad?
   (b) If the ad is allocated using the Vickrey, Clarke, Groves mechanism, what would be the associated payments? Would the seller get the same expected revenues as with the auction?
   (c) Does an efficient, incentive compatible and individually rational mechanism exist in this case that also exactly balances the seller’s budget? What mechanism would you suggest for this purpose?

3. Consider a common value auction with two bidders. Each bidder gets a signal on the value of the object. Signals are independently and uniformly distributed on the interval \([0, 1]\). The true value of the object, unknown to the bidders before the auction, is given by \(V = 2x_1 - x_2\).
   (a) Find the equilibrium bidding strategies for the first- and second-price auctions.
   (b) Compute the associated equilibrium expected revenue for each of the two auction formats.